Strictly Based on Latest Syllabus Issued by CBSE
for
2014 Examination

Sample Question Papers

Solutions

Mathematics

Class **12**

OSWAAL BOOKS

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Sample Question Paper—6

Section–A

1. Given $a * b = \text{HCF}(a,b);\ a,b \in \mathbb{N}$

then $22 * 4 = \text{HCF of 22 and 4}$

$= 2$ **1**

2. $\sin^{-1}\left(\sin\frac{3\pi}{5}\right)$

$\sin^{-1}\left(\sin\frac{3\pi}{5}\right) \neq \frac{3\pi}{5}$, as $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$\sin^{-1}\left(\sin\frac{3\pi}{5}\right) = \sin^{-1}\left(\sin\left(\pi - \frac{2\pi}{5}\right)\right)$

$= \sin^{-1}\left(\sin\frac{2\pi}{5}\right)$ **½**

$= \frac{2\pi}{5}$ **½**

3. $\begin{vmatrix} 3x+y & -y \\ 2y-x & 3 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ -5 & 3 \end{vmatrix}$

We have $3x + y = 1$ ...(i) **½**

and $-y = 2$

or $y = -2$

From equation (i),

$3x + (-2) = 1$

$\Rightarrow 3x = 3$

$\Rightarrow x = 1$ **½**

4. Given, $|A| = 5$, order of matrix $(n) = 3$

$|\text{adj } A| = |A|^{n-1}$

$\Rightarrow |\text{adj } A| = 5^{3-1}$

or $|\text{adj } A| = 25$ **1**

5. Given, $\begin{vmatrix} x & 4 \\ 2 & 2x \end{vmatrix} = 0$

$\Rightarrow x \times 2x - 4(2) = 0$

$\Rightarrow 2x^2 - 8 = 0$

$\Rightarrow 2x^2 = 8$ **½**

$x^2 = 4$

$\Rightarrow x = \pm 2$ **½**

6. $\int \frac{\sec^2 x}{3 + \tan x}\,dx$

Put $3 + \tan x = t$

Then $\sec^2 x\,dx = dt$ **½**

$\Rightarrow \int \frac{1}{t}\,dt = \log|t| + c$

$= \log|(3 + \tan x)| + c$ **½**

7. $\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$

Put $\sqrt{x} = t$, then **½**

$\frac{1}{2\sqrt{x}}\,dx = dt$

$= -2\cos t + c$

$= -2\cos\sqrt{x} + c$ **½**

8. $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 6 & 27 \\ 1 & 3 & P \end{vmatrix} = 0$

$\Rightarrow \hat{i}(6 \times P - 27 \times 3) - \hat{j}(2 \times P - 27 \times 1) + \hat{k}(2 \times 3 - 6 \times 1) = 0$ **½**

$\Rightarrow \hat{i}(6P - 81) - \hat{j}(2P - 27) + \hat{k}(6 - 6) = 0$

$\Rightarrow \hat{i}(6P - 81) - \hat{j}(2P - 27) + 0\hat{k}$

$= 0\hat{i} + 0\hat{j} + 0\hat{k}$

Comparing the coefficients of $\hat{i}$ and $\hat{j}$ from both sides, we get

$6P - 81 = 0$

and $2P - 27 = 0$

$\Rightarrow P = \frac{27}{2}$ **½**

9. Given, $(\vec{x} - \vec{p}) \cdot (\vec{x} + \vec{p}) = 80$

$\Rightarrow \vec{x} \cdot \vec{x} + \vec{x} \cdot \vec{p} - \vec{p} \cdot \vec{x} - \vec{p} \cdot \vec{p} = 80$

$\Rightarrow |\vec{x}|^2 - |\vec{p}|^2 = 80$, as $\vec{x} \cdot \vec{p} = \vec{p} \cdot \vec{x}$

$\Rightarrow |\vec{x}|^2 - 1 = 80$, as $|\vec{p}| = 1$ **½**

$\Rightarrow \quad |\vec{x}|^2 = 81$

$\therefore \quad |\vec{x}| = 9$ **½**

10. $\vec{a} = 3\hat{i} + 2\hat{j} + 9\hat{k}$, $\vec{b} = \hat{i} + p\hat{j} + 3\hat{k}$

There $a_1 = 3, b_1, = 2, c_1 = 9$

$a_2 = 1, b_2 = p, c_2 = 3$

$\vec{a}$ and $\vec{b}$ are parallel, if $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{c_1}{c_1}$ **½**

$\Rightarrow \quad \frac{3}{p} = \frac{2}{p} = \frac{9}{.3}$

$\Rightarrow \quad p = \frac{2}{3}$ **½**

Section B

11. Let O be the origin, then the relation

R = {(P, Q) : OP = OQ}

(i) R is reflexive. Take any distance OP. We know OP = OP.

$\therefore$ R is reflexive. **1**

(ii) R is symmetric. If OP = OQ, then OQ = OP $\Rightarrow$ R is symmetric. **1**

(iii) R is transitive

Let OP = OQ and OQ = OR $\Rightarrow$ OP = OR.

$\therefore$ R is transitive.

Hence R is an equivalence relation.

Since OP = k (constant) $\Rightarrow$ P lies on a circle with centre at the origin.

12. We have

$$1 + \sin x = \cos^2\frac{x}{2} + \sin^2\frac{x}{2} + 2\cos\frac{x}{2}\sin\frac{x}{2}$$

$$= \left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)^2 \quad \textbf{1}$$

Similarly,

$$1 - \sin x = \left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)^2 \quad \textbf{1}$$

Now, $\cot^{-1}\left(\frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}}\right)$

$$= \cot^{-1}\left(\frac{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right) + \left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)}{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right) - \left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)}\right)$$

$$= \cot^{-1}\left(\frac{2\cos\frac{x}{2}}{2\sin\frac{x}{2}}\right)$$

$$= \cot^{-1}\left(\cot\frac{x}{2}\right) \quad 1$$

$$= \frac{x}{2}$$

OR

$$\int \frac{(x-4)e^x}{(x-2)^3}\,dx$$

Let $\frac{x-4}{(x-2)^3} = \frac{A}{(x-2)} + \frac{B}{(x-2)^2} + \frac{C}{(x-3)^3}$ **1**

or $x - 4 = A(x-2)^2 + B(x-2) + C$

Put $x = 2$, we get C = –2

Comparing coefficients of x^2 and x of both sides, we get

A = 0

and A + B = 1 **1**

$\Rightarrow$ B = 1

$$\therefore \int \frac{(x-4)e^x}{(x-2)^3}dx = \int \left(\frac{1}{(x-2)^2} - \frac{2}{(x-3)^3}\right) e^x\,dx$$

$$= \int \frac{e^x}{(x-2)^2}dx - \int \frac{2e^x dx}{(x-2)^3} \quad \textbf{½}$$

Integrating Ist term by parts

$$\frac{1}{(x-2)^2}e^x - \int \frac{-2}{(x-2)^3}\cdot e^x\,dx + c - \int \frac{2e^x dx}{(x-2)^3}$$

½

$$= \frac{e^x}{(x-2)^2} + \frac{2e^x}{(x-2)^3}\,dx + c - \int \frac{2e^x dx}{(x-2)^3} \quad \textbf{1}$$

$$= \frac{e^x}{(x-2)^2} + c$$

13. **(i)** We have

$$B = \begin{bmatrix} 1000 & 500 & 5000 \\ 3000 & 1000 & 10000 \end{bmatrix}$$

$$A = \begin{bmatrix} 40 \\ 100 \\ 50 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1000 & 500 & 5000 \\ 3000 & 1000 & 10000 \end{bmatrix} \begin{bmatrix} 40 \\ 100 \\ 50 \end{bmatrix}$$

$$= \begin{array}{c} \begin{bmatrix} 40000 & +50000 & +250000 \\ 120000 & +1000000 & +500000 \end{bmatrix} \end{array} \begin{array}{l} \to x \\ \to y \end{array}$$

$$= \begin{array}{c} \begin{bmatrix} 340000 \\ 720000 \end{bmatrix} \end{array} \begin{array}{l} \to x \\ \to y \end{array}$$

So the total amount spent by the group in the two cities is 340000 paise and 720000 paise, *i.e.,* Rs. 3400 and Rs. 7,200 respectively. **1**

(ii) Matrices **1**

(iii) Some idealism is necessary in successful polities. **1**

OR

If α, β, γ are in A.P., then

$$2\beta = \alpha + \gamma$$

Let $\Delta = \begin{bmatrix} x-3 & x-4 & x-\alpha \\ x-2 & x-3 & x-\beta \\ x-1 & x-2 & x-\gamma \end{bmatrix}$

$(R_1 \to R_1 + R_3)$

$\therefore \quad \Delta = \begin{bmatrix} 2x-4 & 2x-6 & 2x-(\alpha+\gamma) \\ x-2 & x-3 & x-\beta \\ x-1 & x-2 & x-\gamma \end{bmatrix}$ **1**

$\therefore \quad \Delta = 2\begin{bmatrix} x-2 & x-3 & x-\beta \\ x-2 & x-3 & x-\beta \\ x-1 & x-2 & x-\gamma \end{bmatrix}$ **1**

$= 2 \times 0$, (two rows are identical) **1**

$\therefore \quad \Delta = 0.$

14. $f(x) = \begin{cases} 2x+1; & x < 2 \\ k; & x = 2 \\ 3x-1; & x > 2 \end{cases}$

Since the function $f(x)$ is continuous at $x = 2$, then

$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^-} f(x)$

$= f(2) = k$ ...(i) **1**

i.e., $\lim_{x\to 2^+} f(x) = \lim_{x\to 0} f(2+x)$

$= \lim_{x\to 0} 2(2+x) + 1$

$= 2(2+0) + 1$ **2**

$= 5$ ...(ii)

From (i) and (ii), $k = 5$ **1**

15. $y = \log \sqrt{\dfrac{1-\cos 2x}{1+\cos 2x}}$

$y = \log \sqrt{\dfrac{1-(1-2\sin^2 x)}{1+2\cos^2 x - 1}}$ **1**

$y = \log \sqrt{\dfrac{\sin^2 x}{\cos^2 x}}$

$y = \log \tan x$

$\dfrac{dy}{dx} = \dfrac{1}{\tan x} \cdot \sec^2 x$ **1**

$\dfrac{dy}{dx} = \dfrac{\cos x}{\sin x} \cdot \dfrac{1}{\cos^2 x}$ **1**

$\dfrac{dy}{dx} = \dfrac{1 \times 2}{2\sin x \cos x}$ **1**

$\dfrac{dy}{dx} = 2 \operatorname{cosec} 2x,$

$(\because 2\sin x \cos x = \sin 2x)$ **1**

Proved.

16. $y = \sin(3x+2)$

$\Rightarrow \quad y + \delta y = \sin[3(x+\delta x)+2]$

$\therefore \quad \delta y = \sin[3(x+\delta x)+2] - \sin(3x+2)$

$= 2\cos\left(\dfrac{3(x+\delta x)+2+(3x+2)}{2}\right)$

$\times \sin\left(\dfrac{3(x+\delta x)+2-(3x+2)}{2}\right)$ **1**

$= 2\cos\left(3x+2+\dfrac{3}{2}\delta x\right) \sin\left(\dfrac{3}{2}\delta x\right)$

$\Rightarrow \quad \dfrac{\delta y}{\delta x} = \cos\left(3x+2+\dfrac{3}{2}\delta x\right) .3. \dfrac{\sin\left(\frac{3}{2}\delta x\right)}{\frac{3}{2}\delta x}$ **1**

Proceeding to limits as $\delta x \to 0$, we get

$\dfrac{dy}{dx} = 3\cos(3x+2).$ **1**

17. Let $t = \int \dfrac{x^2+1}{x^4+1}dx = \int \dfrac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}dx$

$= \int \dfrac{1+\frac{1}{x^2}}{\left(x^2-\frac{1}{x^2}\right)+2}dx$ **1**

Dividing Numerator and Denominator by x^2.

Substituting $x - \dfrac{1}{x} = t \Rightarrow \left(x+\dfrac{1}{x^2}\right) dx = dt$

$I = \int \dfrac{dt}{t+2} = \int \dfrac{dt}{t^2+(\sqrt{2})^2}$ **1**

$$= \frac{1}{\sqrt{2}} \tan^{-1} \frac{t}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{x^2 - \frac{1}{x}}{\sqrt{2}} \right) \quad \textbf{1}$$

$$= \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{x^2 - 1}{\sqrt{2}x} \right) + c. \quad \textbf{1}$$

18. $(x^2 - y^2)\, dx + 2xy\, dy = 0$

$(x^2 - y^2)\, dx = -2xy\, dy$

$$\frac{dy}{dx} = -\left(\frac{x^2 - y^2}{2xy} \right)$$

$$\frac{dy}{dx} = \frac{y^2 - x^2}{2xy} \quad \text{...(i)}$$

Put $y = vx$

$$\frac{dy}{dx} = v + x \frac{dv}{dx} \quad \textbf{1}$$

Putting in (i)

$$v + x \frac{dv}{dx} = \frac{v^2x^2 - x^2}{2x.vx}$$

$$v + x \frac{dv}{dx} = \frac{v^2 - 1}{2v}$$

$$x \frac{dv}{dx} = \frac{v^2 - 1}{2v} - v \quad \textbf{1}$$

$$x \frac{dv}{dx} = -\left(\frac{v^2 - 1}{2v} \right)$$

$$\left(\frac{2v}{v^2 + 1} \right) dv = -\frac{dx}{x}$$

On integrating,

$$\int \frac{2v}{v^2 + 1} dv = -\int \frac{dx}{x}$$

$$\log |v^2 + 1| = -\log |x| + c$$

$$\log |v^2 + 1| + \log |x| = c \quad \textbf{1}$$

$$\log |(v^2 + 1)x| = \log c$$

$$(v^2 + 1)x = c$$

$$\left(\frac{y^2}{x^2} + 1 \right) x = c$$

$$(x^2 + y^2) = cx$$

Putting $x = 1$ and $y = 1$, we get

$$c = 2$$

$$\therefore \quad x^2 + y^2 = 2x. \quad \textbf{1}$$

OR

$$\frac{dy}{dx} + y = \cos x - \sin x \quad \text{...(i)}$$

This is a linear differential equation.

Here $P = 1$, $Q = (\cos x - \sin x)$ **1**

$$\text{I.F.} = e^{\int P\, dx} = e^{\int 1\, dx} = e^x$$

The solution of (1) is

$$y \times \text{I.F.} = \int Q \times \text{I.F.}\, dx + c$$

$$\Rightarrow \quad ye^x = \int (\cos x - \sin x)\, e^x\, dx + c \quad \textbf{1}$$

Put $e^x \cos x = t$

$[e^x . \cos x + e^x .(-\sin x)]\, dx = dt$

or $e^x (\cos x - \sin x)\, dx = dt$

$$\Rightarrow \quad ye^x = \int dt + c \quad \textbf{1}$$

or $ye^x = t + c$

$ye^x = e^x \cos x + c$

or $y = \cos x + ce^{-x}$ **1**

19. $\frac{dy}{dx} = \log (x + 1)$

$dy = \log (x + 1)\, dx$

Integrating both sides **1**

$$\int dy = \int \log (x + 1)\, dx + c$$

$$y = x \log (x + 1) - \int \left(\frac{x}{x + 1} \right) dx + c \quad \textbf{1}$$

$$y = x \log (x + 1) - \int \left(1 dx - \frac{1}{x + 1} \right) dx + c \quad \textbf{1}$$

$$y = x \log (x + 1) - x + \log |x + 1| + c. \quad \textbf{1}$$

20. $\frac{1 - x}{3} = \frac{7y - 14}{2\lambda} = \frac{5z - 10}{11}$

or $\frac{x - 1}{-3} = \frac{y - 2}{\frac{2}{7}\lambda} = \frac{z - 2}{\frac{11}{5}}$ ...(i)

and $\frac{x - 1}{-\frac{3}{7}\lambda} = \frac{y - 5}{1} = \frac{z - 6}{-5}$ ...(ii) **1**

Direction cosines of the given lines are perpendicular, if

$$a_1b_1 + a_2b_2 + c_1c_2 = 0$$

Here $a_1 = -3$, $b_1 = \frac{2}{7}\lambda$, $c_1 = \frac{11}{5}$ **1**

$a_2 = -\frac{3}{7}\lambda$, $b_2 = 1$, $c_1 = -5$

[direction cosines of the lines]

$$\Rightarrow \quad -3 \times \frac{2}{7}\lambda + \left(-\frac{3}{7}\lambda + 1\right) + \left(\frac{11}{5} \times (-5)\right) = 0$$

$$-\frac{6\lambda}{7} - \frac{3\lambda}{7} - 11 = 0 \qquad \textbf{1}$$

$$\Rightarrow \quad -6\lambda - 3\lambda - 77 = 0$$

$$-9\lambda = 77$$

$$\lambda = -\frac{77}{9} \qquad \textbf{1}$$

OR

$$\vec{r} = (1+\lambda)\hat{i} + (2-\lambda)\hat{j} + (\lambda+1).7\hat{k};$$

$$\vec{r} = (2\hat{i} - \hat{j} - \hat{k}) + \mu(2\hat{i} + \hat{j} + 2\hat{k})$$

The given lines are

$$\vec{r} = (\hat{i} + 2\hat{j} + \hat{k}) + \lambda(\hat{i} - \hat{j} + \hat{k}) \qquad \textbf{1}$$

$$\vec{r} = (2\hat{i} - \hat{j} - \hat{k}) + \mu(2\hat{i} + \hat{j} + 2\hat{k})$$

Here $a_1 = \hat{i} + 2\hat{j} + \hat{k}, b_1 = \hat{i} - \hat{j} + \hat{k}$

$$a_2 = 2\hat{i} + \hat{j} + \hat{k}, b_2 = 2\hat{i} + \hat{j} + 2\hat{k} \qquad \textbf{1}$$

$$\vec{a_2} - \vec{a_1} = (2\hat{i} - \hat{j} - \hat{k})) - (\hat{i} + 2\hat{j} + \hat{k}))$$

$$= \hat{i} - 3\hat{j} - 2\hat{k}$$

$$\vec{b_1} \times \vec{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 1 \\ 2 & 1 & 2 \end{vmatrix}$$

$$= \hat{i}[-1 \times 2 - 1 \times 1] - \hat{j}[1 \times 2 - 1 \times 2]$$

$$+ \hat{k}[1 \times 1 - 2 \times (-1)]$$

$$\vec{b_1} \times \vec{b_2} = -3\hat{i} + 3\hat{k}$$

$$|\vec{b_1} \times \vec{b_2}| = \sqrt{(-3)^2 + (3)^2}$$

$$= \sqrt{9+9} = \sqrt{18}$$

The shortest distance between the given lines

$$= \left|\frac{(\vec{b_1} \times \vec{b_2}).(\vec{a_2} - \vec{a_1})}{|\vec{b_1} \times \vec{b_2}|}\right| \qquad \textbf{1}$$

$$= \frac{|(-3\hat{i} + 3\hat{k}).(\hat{i} - 3\hat{j} - 2\hat{k})|}{\sqrt{18}}$$

$$= \frac{|-3 - 0 - 6|}{\sqrt{18}}$$

$$= \frac{9}{\sqrt{18}}$$

$$= \frac{9}{3\sqrt{2}}$$

$$= \frac{3}{\sqrt{2}}$$

$$= \frac{3\sqrt{2}}{2} \qquad \textbf{1}$$

21. With the usual notation, we have

$n = 3$

p = probability that student will graduate

$= 0.4$

$q = 1 - p = 1 - 0.4 = 0.6$

$P(r)$ = probability that out of 3 students r will graduate

$= {}^3C_r (0.4)^r (0.6)^{3-r}$

$r = 0, 1, 2, 3.$ **1**

(i) Required probability

$= P(0) = {}^3C_0 (0.4)^0 (0.6)^3$

$= 1 \times 1 \times (0.6)^3 = 0.216$ **1**

(ii) Required probability

$= P(1) = {}^3C_1 (0.4)^1 (0.6)^2$

$= 3 \times 0.4 \times 0.36 = 0.432$. **1**

(iii) Required probability

$= P(3) = {}^3C_3 (0.4)^3 (0.6)^0$

$= 1 \times 0.064 \times 1 = 0.064$

1

22. Let E_1 and E_2 be the events that the insured vehicle is a scooter and a motor cycle respectively. Let A be the event that an insured vehicle meets an accident, then

$$P(E_1) = \frac{2{,}000}{2{,}000 + 3{,}000} = \frac{2}{5}$$

and $$P(E_2) = \frac{3{,}000}{2{,}000 + 3{,}000} = \frac{3}{5} \qquad \textbf{1}$$

Also $$P\left(\frac{A}{E_1}\right) = 0.01 = \frac{1}{100}$$

and $$P\left(\frac{A}{E_2}\right) = 0.02 = \frac{2}{100} \qquad \textbf{1}$$

By Baye's theorem, the required probability

$$P\left(\frac{E_2}{A}\right) = \frac{P(E_2) \times P\left(\frac{A}{E_2}\right)}{P(E_1) \times P\left(\frac{A}{E_1}\right) + P(E_2) \times P\left(\frac{A}{E_2}\right)}$$

1

$$= \frac{\frac{3}{5}\times\frac{2}{100}}{\frac{2}{5}\times\frac{1}{100}+\frac{3}{5}\times\frac{2}{100}}$$

$$= \frac{6}{2+6} = \frac{6}{8} = \frac{3}{4} = 0.75. \qquad \textbf{1}$$

Section C

23. Given,

$$A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$$

$$|A| = 2(-4+4) + 3(-6+4) + 5(3-2)$$
$$= 0 - 6 + 5$$
$$= -1 \neq 0 \qquad \textbf{1}$$

$\therefore$ A^{-1} exists and it is given by

$$A^{-1} = \frac{1}{|A|}\text{Adj } A = \frac{1}{-1}\begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix}$$

Now, given system of equations can be written as

$$A_1 X = B$$

where $$A_1 = \begin{bmatrix} 2 & -3 & 5 \\ 3 & -2 & -4 \\ 1 & -1 & -2 \end{bmatrix} \qquad \textbf{1}$$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 16 \\ -4 \\ -3 \end{bmatrix}$$

Clearly $A_1 = A \Rightarrow A_1^{-1} = A^{-1}$ **1**

$\therefore$ by (1),

$$X = A_1^{-1} B$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{-1}\begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix}\begin{bmatrix} 16 \\ -4 \\ -3 \end{bmatrix} \qquad \textbf{1}$$

$$= -1\begin{bmatrix} 0+4-6 \\ 32+36-69 \\ 16+20-39 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

$\therefore$ $x = 2, y = 1$ and $z = 3$. **1**

24. Let the line through the given point (3,4) be

$$\frac{x}{a} + \frac{y}{b} = 1 \qquad ...(i)$$

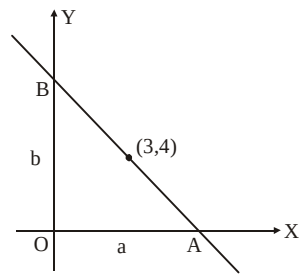
Then, the point (3, 4) lies on line (i), therefore

$$\frac{3}{a} + \frac{4}{b} = 1 \qquad ...(ii)$$

Clearly, OA = a and OB = b.

Let Δ be the area of triangle OAB, then

$$\Delta = \frac{1}{2} \times OA \times OB = \frac{1}{2}ab \qquad ...(iii)$$



By (ii), $$\frac{4}{b} = 1 - \frac{3}{a} = \frac{a-3}{a} \qquad \textbf{1}$$

$$\Rightarrow \quad b = \frac{4a}{a-3} \qquad ...(iv)$$

Eliminating b from (iii) and (iv), we get

$$\Delta = \frac{2a^2}{a-3}$$

Now, $$\frac{d\Delta}{da} = 2\left[\frac{(a-3)\times 2a - a^2(1)}{(a-3)^2}\right] \qquad \textbf{1}$$

$$= 2\left[\frac{a^2-6a}{(a-3)^2}\right]$$

and $$\frac{d^2\Delta}{da^2} = \left[\frac{(a-3)^2(2a-6)-(a^2-6a).2(a-3)}{(a-3)^4}\right]$$

For maximum or minimum of Δ,

$$\frac{d\Delta}{da} = 0$$

i.e., $a^2 - 6a = 0$

$\Rightarrow$ $a(a-6) = 0$

$\Rightarrow$ $a = 0$ or $a = 6$ **1**

Since $a \neq 0$,

$$a = 6$$

When $a = 6$, $$\frac{d^2\Delta}{da^2} = 2\left[\frac{9\times 6 - 0}{81}\right] = \frac{4}{3} > 0.$$

$\therefore$ Δ is minimum when $a = 6$ and also by (iv), **1**

$$b = \frac{4\times 6}{6-3} = \frac{24}{3} = 8.$$

Consequently by (i), we get $\frac{x}{6} + \frac{y}{8} = 1$.

This is the required equation of the line through the point (3, 4). **1**

OR

Let h be the height and r be the radius of the inscribed cylinder. Let V be the volume of the cylinder, then

$$V = \pi r^2 h \quad \text{...(i)} \quad \mathbf{1}$$

Given, radius OA of the sphere = $5\sqrt{3}$ cm.

$\therefore$ In ΔOEA,

$$r^2 = (5\sqrt{3})^2 - \left(\frac{h}{2}\right)^2$$

$$\Rightarrow \quad r^2 = 75 - \frac{h^2}{4} \quad \mathbf{1}$$

From (i) and (ii), we have

$$V = \pi\left(75 - \frac{h^2}{4}\right)h$$

$$= \pi\left(75h - \frac{h^3}{4}\right)$$

$$\therefore \quad \frac{dV}{dh} = \pi\left(75 - \frac{3h^2}{4}\right) \quad \mathbf{1}$$

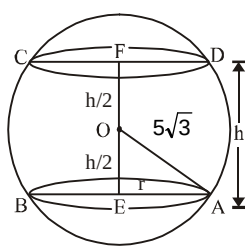
and $$\frac{d^2V}{db^2} = \pi\left(-\frac{3h}{2}\right)$$

For maximum or minimum of V,

$$\frac{dV}{dh} = 0$$

i.e., $$75 - \frac{3h^2}{4} = 0$$

$$\Rightarrow \quad h^2 = \frac{75 \times 4}{3} = 100 \quad \mathbf{1}$$



$$\therefore \quad h = 10.$$

At $h = 10$, $\dfrac{d^2V}{dh^2} = \pi\left(-\dfrac{3 \times 10}{2}\right) = -15\pi < 0$ **1**

Thus, V is maximum when $h = 10$.

Putting $h = 10$ in (ii), we get

$$r^2 = 75 - \frac{100}{4} = 50$$

$$\Rightarrow \quad r = 5\sqrt{2}$$

Putting $r = 5\sqrt{2}$ and $h = 10$ in (i), the maximum volume of the cylinder is given by

$$V_{max} = \pi\left(5\sqrt{2}\right)^2 \times 10$$

$$= 500\pi \text{ cm}^3. \quad \mathbf{1}$$

25. Here the curves are

$$x^2 + y^2 = 16 \quad \text{...(1)}$$

$$y^2 = 6x \quad \text{...(2)}$$

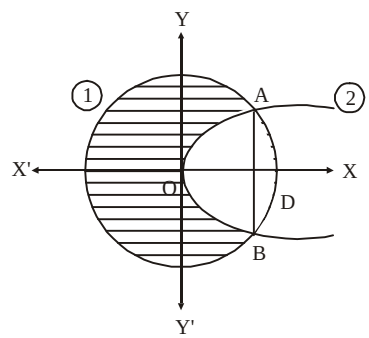
Equations (1) and (2) intersect at A and B. Their x-coordinates are given by

$$x^2 + 6x - 16 = 0$$

$$(x + 8)(x - 2) = 0$$

$$x = -8, 2 \quad \mathbf{1}$$

Rejecting $x = 8$, so $x = 2$



Also C is (–4, 0)

$\therefore$ The required area

= 2 × Area OACO

= 2 [Area ODACOD – Area ODAO]

$$= 2\left[\int_{-4}^{2}\sqrt{16 - x^2}\,dx - \int_0^2 \sqrt{6x}\,dx\right]$$

$$\times \left[\left[\frac{x}{2}\sqrt{16 - x^2} + \frac{16}{2}\sin^{-1}\frac{x}{4}\right]_0^2 - \sqrt{6}\left[\frac{x^{3/2}}{3/2}\right]_0^2\right]$$

$$= 2\left[\frac{2}{2}\sqrt{16 - 4} + 8\sin^{-1}\frac{1}{2}\right.$$

$$\left. - \left(\frac{-4}{2}\sqrt{16 - 16} + 8\sin^{-1}\frac{-4}{4}\right)\right.$$

$$\left. - \sqrt{6}\left(\frac{2}{3} \times (2)^{3-2} - 0\right)\right]$$

$$= 2\left[\sqrt{12} + 8 \times \frac{\pi}{6} - (0 - 8\sin^{-1}1) - \sqrt{6} \times \frac{4\sqrt{2}}{3}\right]$$

$$= 2\left[2\sqrt{3} + \frac{4}{3}\pi + 8 \times \frac{\pi}{2} - \frac{8\sqrt{3}}{3}\right]$$

$$= 2\left[\frac{16\pi - 2\sqrt{3}}{3}\right] = \frac{32\pi - 4\sqrt{3}}{3} \text{ sq. units.} \quad \mathbf{1}$$

OR

Let 'r' be the radius of base, 'h' the height and 'l' the slant height of the cone whose volume is V and curved surface C.

Given, $$V = \frac{1}{3}\pi r^2 h \quad ...(i)$$

Also, $$l^2 = h^2 + r^2 \quad ...(ii)$$

$$C = \pi r l$$

$$\Rightarrow \quad C^2 = \pi^2 r^2 l^2 \quad \textbf{1}$$

If C is minimum, C^2 is also minimum

$$C^2 = D = \pi^2 r^2 (h^2 + r^2)$$

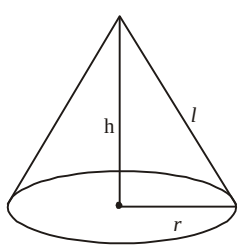
$$\Rightarrow \quad D = \pi^2 r^2 \left(\frac{9V^2}{\pi^2 r^4} + r^2\right)$$

$$\Rightarrow \quad D = \frac{9V^2}{r^2} + \pi^2 r^4 \quad \textbf{1}$$

$$\Rightarrow \quad \frac{dD}{dr} = \frac{-18V^2}{r^3} + 4\pi^2 r^3$$

For minimum D,

$$\frac{dD}{dr} = 0$$



1

$$\Rightarrow \quad \frac{-18V^2}{r^3} + 4\pi^2 r^3 = 0$$

$$\Rightarrow \quad 4\pi^2 r^6 = 18V^2$$

$$\Rightarrow \quad r = \left(\frac{9V^2}{2\pi^2}\right)$$

$$\frac{d^2D}{dr^2} = \frac{54V^2}{r^4} + 12\pi^2 r^2$$

$$\left(\frac{d^2D}{dr^2}\right)_{r = \left(\frac{9V^2}{2\pi^2}\right)^{1/6}} > 0$$

$\therefore$ Curved area is least for,

$$r = \left(\frac{9V^2}{2\pi^2}\right)^{1/6}$$

$$\Rightarrow \quad 2\pi^2 r^6 = 9V^2$$

$$\Rightarrow \quad 2\pi^2 r^6 = \pi^2 r^4 h^2, \quad \text{[From (i)]} \ \textbf{1}$$

$$\Rightarrow \quad 2\pi^2 = h^2$$

$$\Rightarrow \quad h = \sqrt{2}\, r$$

Hence, curved area is minimum, when height of the cone is $\sqrt{2}$ times the radius of the base of the cone. **1**

26. Let $x = a - t \Rightarrow dx = -dt$

When $x = 0$,

$t = a$ and when $x = a$, $t = 0$

$$\therefore \quad \int_0^a f(x)\,dx = \int_a^0 f(a-t)\,(-dt)$$

$$= -\int_a^0 f(a-t)\,dt \quad \textbf{1}$$

$$= \int_0^a f(a-x)\,dx,$$

[By property of definite integral the variable t to x]

$$= \int_0^a f(a-t)\,dt, \quad \textbf{1}$$

[By changing the variable t to x]

This proves the first part of question.

Now using above property, we have

$$\int_0^2 x\sqrt{2-x}\,dx = \int_0^2 (2-x)\sqrt{2-(2-x)}\,dx \quad \textbf{1}$$

$$= \int_0^2 (2-x)\sqrt{x}\,dx$$

$$= \int_0^2 (2x^{1/2} - x^{3/2})\,dx$$

$$= \left[2\frac{x^{3/2}}{3/2} - \frac{x^{5/2}}{5/2}\right]_0^2 \quad \textbf{1}$$

$$= \left[\frac{4}{3}x^{3/2} - \frac{2}{5}x^{5/2}\right]_0^2$$

$$= \left[\frac{4}{3}\times 2^{3/2} - \frac{2}{5}\times 2^{5/2}\right] - (0-0) \quad \textbf{1}$$

$$= \frac{8\sqrt{2}}{3} - \frac{8\sqrt{2}}{5}$$

$$= \frac{16\sqrt{2}}{15} \quad \textbf{1}$$

27. The two given planes are

$$\vec{r}\cdot(\hat{i} + 2\hat{j} + 3\hat{k}) - 4 = 0 \quad ...(1)$$

$$\vec{r}\cdot(2\hat{i} + \hat{j} - \hat{k}) + 5 = 0 \quad ...(2) \ \textbf{1}$$

A plane which contains the line of intersection of the planes (1) and (2) is

$$\vec{r}\cdot(\hat{i} + 2\hat{j} + 3\hat{k}) - 4 + \lambda[\vec{r}\cdot(2\hat{i} + \hat{j} - \hat{k}) + 5] = 0$$

or, $\vec{r}.(1 + 2\lambda)\hat{i} + (2 + \lambda)\hat{j} + (3 - \lambda)\hat{k}] - 4 + 5\lambda = 0$ ...(3)

Now the plane (3) is perpendicular to the plane

$\vec{r}.(5\hat{i} + 3\hat{j} - 6\hat{k}) + 8 = 0$

$\Rightarrow$ $(1 + 2\lambda) \times 5 + (2 + \lambda) \times 3 + (3 + \lambda) \times (-6) = 0$ **1**

or, $5 + 10\lambda + 6 + 3\lambda - 18 + 6\lambda = 0$

$$19\lambda - 7 = 10 \Rightarrow \lambda = \frac{7}{19}$$ **1**

Putting the value of λ in (3),

$$\vec{r}.\left[\left(1 + \frac{14}{19}\right)\hat{i} + \left(2 + \frac{7}{19}\right)\hat{j} + \left(3 - \frac{7}{19}\right)\hat{k}\right] - 4 + 5 \times \frac{7}{19} = 0$$ **1**

or $\vec{r}.\left[\frac{33}{19}\hat{i} + \frac{45}{19}\hat{j} + \frac{50}{19}\hat{k}\right] + \frac{-76 + 35}{19} = 0$

or $\vec{r}.(33\hat{i} + 45\hat{j} + 50\hat{k}) - 41 = 0$

This is the required plane. **1**

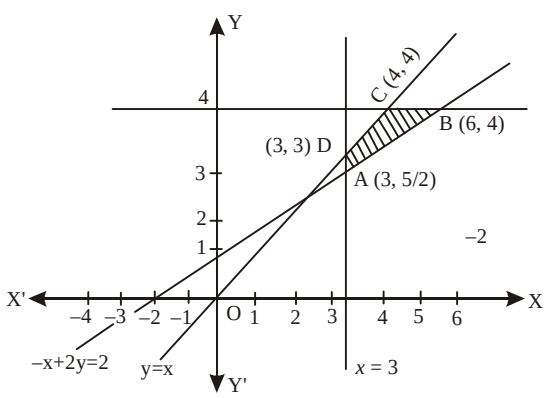
28. Draw the lines $x - y = 0$, $-x + 2y = 2$, $x = 3$ and $y = 4$ with $x, y \geq 0$.

The shaded region ABCD in the figure is feasible region of the given L.P.P. The corner points of the feasible region are A $\left(3, \frac{5}{2}\right)$, B (6, 4), C(4, 4) and D (3, 3). **1**

Now, value of $z = x - 5y + 20$

At A $\left(3, \frac{5}{2}\right)$, $z = 3 - 5 \times \frac{5}{2} + 20 = 10.5$ **1**

At B (6, 4), $z = 6 - 5 \times 4 + 20 = 6$



At C (4, 4), $z = 4 - 5 \times 4 + 20 = 4$ **1**

At D (3, 3), $z = 3 - 5 \times 3 + 20 = 8$ **1**

Clearly, minimum value of z is 4 at C(4, 4) *i.e.*, when $x = 4$ and $y = 4$ **1**

29. Bag I contains 3 red and 4 black balls.

Bag II contains 4 red and 5 black balls.

Let E_1 = Event that a red ball is drawn from Bag I

E_2 = Event that a black ball is drawn from Bag **1**

$\therefore$ $P(E_1) = \frac{3}{7}$, $P(E_2) = \frac{7}{4}$ **1**

After transferring a red ball from bag I to bag II, the bag I will have 5 red and 5 black balls.

Let A be the event of drawing red ball

$\therefore$ $P(A/E_1) = \frac{5}{10} = \frac{1}{2}$ **1**

Further when a black ball is transferred from bag I to bag II, it will contain 4 red and 6 black balls.

$$P(A/E_2) = \frac{4}{10} = \frac{2}{5}$$ **1**

$\therefore$ $P(E_2/A) = \dfrac{P(E_2) \times P(A/E_2)}{P(E_1)P(A/E_1) + P(E_2)P(A/E_2)}$ **1**

$$= \frac{\frac{4}{7} \times \frac{2}{5}}{\frac{3}{7} \times \frac{1}{2} + \frac{4}{7} \times \frac{2}{5}}$$

$$= \frac{\frac{8}{35}}{\frac{3}{15} + \frac{8}{35}} = \frac{\frac{8}{35}}{\frac{15 + 16}{70}}$$ **1**

$$= \frac{8}{35} \times \frac{70}{31} = \frac{16}{31}$$ **1**

●●

Sample Question Paper—7

Section–A

1. $f(x) = x + 7$

$g(x) = x - 7$

$fog = f\{g(x)\}$ **½**

$= x - 7 + 7$

$= x$ **½**

$fog\,(7) = 7.$

2. Let $\sin^{-1}\left(-\frac{1}{2}\right) = y$

$\therefore \quad \sin y = -\frac{1}{2} = -\sin\frac{\pi}{6}$

$= \sin\left(-\frac{\pi}{6}\right)$ **½**

The range of principal value branch of $\sin^{-1}$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Hence principal value of $\sin^{-1}\left(-\frac{1}{2}\right)$ is $-\frac{\pi}{6}$. **½**

3. Let $A = \begin{vmatrix} 3-2x & x+1 \\ 2 & 4 \end{vmatrix}$

A is singular matrix

$\Rightarrow \quad |A| = 0$

$\Rightarrow \begin{vmatrix} 3-2x & x+1 \\ 2 & 4 \end{vmatrix} = 0$ **½**

$\Rightarrow 4(3 - 2x) - 2(x + 1) = 0$

$\Rightarrow 12 - 8x - 2x - 2 = 0$

$\Rightarrow 10 - 10x = 10$

$\Rightarrow 10x = 10$

$\Rightarrow x = 1$ **½**

4. (i) $18 = 1 \times 18 = 2 \times 9 = 3 \times 6$

There are 6 matrices of order

$1 \times 18, 18 \times 1, 2 \times 9, 9 \times 2, 3 \times 6, 6 \times 3.$ **½**

(ii) $5 = 1 \times 5$

There are two matrices with 5 elements of order $1 \times 5, 5 \times 1$ **½**

5. $\begin{vmatrix} a+ib & c+id \\ -c+id & a-ib \end{vmatrix}$

$= (a + ib)(a - ib) - (c + id)(-c + id)$ **½**

$= (a^2) - (ib)^2 - [id)^2 - (c)^2]$

$= a^2 + b^2 - [-d^2 - c^2]$

$= a^2 + b^2 + c^2 + d^2$ **½**

6. $I = \int \frac{x^2}{1+x^3}\,dx$

$= \frac{1}{3}\int \frac{3x^2}{1+x^3}\,dx$

$= \frac{1}{3}\log|1 + x^3| + c$ **1**

7. $I = \int \frac{2\cos x}{\sin^2 x}\,dx$

$= \frac{2}{3}\int \frac{\cos x}{\sin x} \times \frac{1}{\sin x}\,dx$

$= \frac{2}{3}\int \cot x.\text{cosec}\,x\,dx$ **½**

$= \frac{2}{3}(-\text{cosec}\,x) + c$

$= -\frac{2}{3}\,\text{coses}\,x + c.$ **½**

8. $\vec{a}.\vec{b} = 8$

$\vec{b} = 2\hat{i} + 6\hat{j} + 3\hat{k}$

Projection of $\vec{a}$ on $\vec{b}$ is given by

$\frac{\vec{a}.\vec{b}}{|b|} = \frac{8}{\sqrt{2^2 + 6^2 + 3^2}}$ **½**

$= \frac{8}{\sqrt{49}}$

$= \frac{8}{7}$ **½**

9. $\vec{a} = 2\hat{i} - 6\hat{j} + 3\hat{k}$

A unit vector in the direction of $\vec{a}$ is

$\frac{\vec{a}}{|\vec{a}|} = \frac{2\hat{i} - 6\hat{j} + 3\hat{k}}{\sqrt{2^2 + 6^2 + 3^2}}$ **½**

$= \frac{2\hat{i} - 6\hat{j} + 3\hat{k}}{\sqrt{49}}$

$$= \frac{2\hat{i}}{7} - \frac{6\hat{j}}{7} + \frac{3\hat{k}}{7}$$ ½

10. Sum of the vectors $2\hat{i} + 4\hat{j} - 5\hat{k}$ and $\lambda\hat{i} + 2\hat{j} + 3\hat{k}$

$= (2\hat{i} + 4\hat{j} - 5\hat{k}) + (\lambda\hat{i} + 2\hat{j} + 3\hat{k})$

$= (2 + \lambda)\hat{i} + 6\hat{j} - 2\hat{k} = s$, say

$$|\vec{s}| = 1(2+\lambda)\hat{i} + 6\hat{j} - 2\hat{k}$$ **1**

$$= \sqrt{(2+\lambda)^2 + 36 + 4}$$

$$= \sqrt{\lambda^2 + 4\lambda + 4 + 36 + 4}$$

$$= \sqrt{\lambda^2 + 4\lambda + 44}$$

$$= \gamma, \text{ say} \quad ...(1)$$

Unit vector $\vec{s} = \frac{2+\lambda}{r}\hat{i} + \frac{6}{r}\hat{j} - \frac{2}{r}\cdot\hat{k}$ ½

The scalar product of $\hat{i} + \hat{j} + \hat{k}$ and unit vector

$$\vec{s} = (\hat{i} + \hat{j} + \hat{k}).\left(\frac{2+\lambda}{r}\hat{i} + \frac{6}{r}\hat{j} - \frac{2}{r}\cdot\hat{k}\right)$$

or $\frac{2+\lambda}{r} + \frac{6}{r} - \frac{2}{r} = 1$

or $\frac{\lambda+6}{r} = 1$

$\lambda + 6 = r$

From (1), $(\lambda + 6)^2 = r^2$

$= \lambda^2 + 4\lambda + 44$

or $\lambda^2 + 12\lambda + 36 = \lambda^2 + 4\lambda + 44$

or $8\lambda = 44 - 36$

$8\lambda = 8$

or $\lambda = 1$ ½

Section–B

11. (i) In a set of triangles

$R = \{(T_1, T_2) : T_1 \text{ is similar to } T_2\}$

(a) A triangle T is similar to itself. **1**

$\therefore$ R is reflexive.

(b) If triangle T_1 is similar to triangle T_2, then T_2 is similar to triangle T_1.

$\therefore$ R is symmetric.

(c) Let T_1 is similar to triangle T_2 and T_2 to T_3, then triangle T_1 is similar to triangle T_3.

$\therefore$ R is transitive.

Hence R is an equivalence relation. **1**

(ii) Two triangles are similar if their sides are proportional. Now sides 3, 4, 5 of triangle T_1 are proportional to the sides 6, 8, 10 of triangle T_3. **1**

Therefore, T_1 is related to T_3.

12. $\sin\left(\frac{\pi}{3} - \sin^{-1}\left(-\frac{1}{2}\right)\right)$

Let $\sin^{-1}\left(-\frac{1}{2}\right) = \theta$ **1**

$\sin\theta = -\frac{1}{2}$

$\sin\theta = -\sin\left(\frac{\pi}{6}\right)$

$\sin\theta = \sin\left(-\frac{\pi}{6}\right)$ **1**

$\theta = -\frac{\pi}{6} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\therefore \sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$

$\sin\left(\frac{\pi}{3} - \sin^{-1}\left(-\frac{\pi}{6}\right)\right)$ **1**

$= \sin\left(\frac{\pi}{3} + \frac{\pi}{6}\right)$

$= \sin[105]$. **1**

13. L.H.S. $= \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}$

$= \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^3 & b^3-a^3 & c^3-a^3 \end{vmatrix}; \begin{matrix} C_2 \to C_2 - C_1 \\ C_3 \to C_3 - C_1 \end{matrix}$ **1**

$= 1\cdot\begin{vmatrix} b-a & c-a \\ b^3-a^3 & c^3-a^3 \end{vmatrix}$

$= (b-a)(c-a)\begin{vmatrix} 1 & 1 \\ b^2+ab+a^2 & c^2+ca+a^2 \end{vmatrix}$ **1**

$= (b-a)(c-a)$

$\times \begin{vmatrix} 1 & 0 \\ b^2+ab+a^2 & (c^2-b^2)+(ca-ab) \end{vmatrix};$

$C_2 \to C_2 - C_1$ **1**

$= (b-a)(c-a)\{(c-b)(c+b) + a(c-b)\}$

$= (b-a)(c-a)(c-b)(c+b+a)$

$= (a-b)(b-c)(c-a)(a+b+c)$

= R.H.S. **1**

OR

$$A = \begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix}$$

Applying $C_1 \to C_1 - C_2$; $C_2 \to C_2 - C_3$

$$A = \begin{vmatrix} 4-x & 0 & 2x \\ x-4 & 4-x & 2x \\ 0 & x-4 & x+4 \end{vmatrix}$$

$$A = (4-x^2)\begin{vmatrix} 1 & 0 & 2x \\ -1 & 1 & 2x \\ 0 & -1 & x+4 \end{vmatrix}$$ **1**

$A = (4-x)^2 [x + 4 + 2x) + 1(2x)]$

$= (4-x)^2 (3x + 4 + 2x)$ **1**

$= (4-x)^2 (5x + 4).$ **1**

14. $$y = \frac{\sin^{-1} x}{\sqrt{1-x^2}}$$

or $$y\sqrt{1-x^2} = \sin^{-1} x$$

Differentiating w.r.t. x we get

$$\sqrt{1-x^2} \cdot \frac{dy}{dx} + y \cdot \frac{1}{2}(1-x^2)^{-1/2}(-2x)$$ **1**

$$= \frac{1}{\sqrt{1-x^2}}$$

or $$(1-x^2)\frac{dy}{dx} - xy = 1$$

Again differentiating w.r.t x, we get

$$(1-x^2)\frac{d^2y}{dx^2} + \frac{dy}{dx}(-2x) - \left(x\frac{dy}{dx} + y1\right) = 0$$ **1**

or $$(1-x^2)\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} - y = 0$$ **1**

15. Let $$y = x^x - 2^{\sin x}$$

$$y = u - v$$

Diff. w.r.t. x

$$\frac{dy}{dx} = \frac{du}{dx} - \frac{dv}{dx}$$ **1**

Now $$u = x^x$$

Taking log of both sides, **1**

$$\log u = \log x^x = x \log x$$

$$[\log m^n = n \log m]$$

Differentiating

$$\frac{1}{u}\frac{du}{dx} = 1.\ \log x + x.\ \frac{1}{x} = (1 + \log x)$$

$$\therefore \quad \frac{du}{dx} = u(1 + \log x) = x^2(1 + \log x)$$

Taking $$u = 2^{\sin x},$$

$$\log u = \log 2^{\sin x} = \sin x \log 2$$ **1**

$$\therefore \quad \frac{1}{u}\frac{du}{dx} = \cos x \log 2$$

$$\therefore \quad \frac{du}{dx} = u \cos x \log 2 = 2^{\sin x} \cos x \log 2$$

$$\therefore \quad \frac{dy}{dx} = \frac{du}{dx} - \frac{du}{dx}$$

$= x^x (1 + \log x) - 2^{\sin x} \cos x \log 2.$ **1**

16. Let $y = (1 + x)(1 + x^2)(1 + x^4)(1 + x^8)$

Taking log of both sides

$\log y = \log (1 + x) + \log (1 + x^2) + \log (1 + x^4) + \log (1 + x^8)$ **1**

Differentiating w.r.t. x

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{1+x} + \frac{1}{1+x^2}\frac{d}{dx}(1+x^2)$$

$$+ \frac{1}{1+x^4}\frac{d}{dx}(1+x^4) + \frac{1}{1+x^8}\frac{d}{dx}(1+x^8)$$ **1**

$$= \frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \frac{8x^7}{1+x^8}$$

$$\therefore \frac{dy}{dx} = y\left(\frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \frac{8x^7}{1+x^8}\right)$$

$= (1 + x)(1 + x^2)(1 + x^4)(1 + x^8)$

$$\times \left(\frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \frac{8x^7}{1+x^8}\right)$$ **1**

At $x = 1$,

$$f(1) = 2.2.2.2\left(\frac{1}{2} + \frac{2}{2} + \frac{4}{2} + \frac{8}{2}\right)$$

$$= 16\left(\frac{1+2+4+8}{2}\right) = \frac{16 \times 15}{2} = 120.$$ **1**

17. Let $$I = \int \frac{\cos(x+a)}{\sin(x+b)}\, dx$$

Let $z = x + b \therefore dz = dx$ **1**

$$I = \int \frac{\cos(z-b+a)}{\sin z}\, dz$$

$$= \int \frac{\cos z \cos(a-b) - \sin z \sin(a-b)}{\sin z}\, dz$$ **1**

$$= \int \cot z \cos(a-b)\, dz - \int \sin(a-b)\, dz$$ **1**

$= \cos(a-b)\log|\sin z| - \sin(a-b)\,z + c$

$= \cos(a-b)\log|\sin(x+b)| - (x+b)\sin(a-b) + c.$ **1**

OR

$$I = \int \sin x\sqrt{1+(2\cos^2 2x - t)}\,dx$$

$$= \int \sin x\sqrt{2}.\cos x\,dx$$

$$= \sqrt{2}\int \sin x.\cos x\,dx \qquad \textbf{1}$$

Let $\sin x = 1$

$\therefore \quad \cos x\,dx = dt$

$$I = \sqrt{2}\int t\,dt + c \qquad \textbf{1}$$

$$= \sqrt{2}\cdot\frac{t^2}{2} + c$$

$$= \frac{t^2}{\sqrt{2}} + c$$

$$= \frac{\sin^2 x}{\sqrt{2}} + c \qquad \textbf{1}$$

18. We have $y = A\cos x - B\sin x$ ...(i)

$$\frac{dy}{dx} = -A\sin x - B\cos x$$

$$\frac{d^2y}{dx^2} = -A\cos x + B\sin x \qquad \textbf{1}$$

or $\dfrac{d^2y}{dx^2} = -(A\cos x - B\sin x)$ **1**

$\therefore \quad \dfrac{d^2y}{dx^2} = -y$ **1**

$\therefore \quad \dfrac{d^2y}{dx^2} + y = 0.$ **Hence proved 1**

OR

$(y^2 - x^2)\,dy = 3xy\,dx$ ...(i)

$$\frac{dy}{dx} = \frac{3xy}{y^2 - x^2}$$

Put $y = ux$

$$\frac{dy}{dx} = u + x\frac{du}{dx} \qquad \textbf{1}$$

From equation (i), we have

$$u + x\frac{du}{dx} = \frac{3x.\nu u}{u^2x^2 - x^2}$$

$$u + x\frac{du}{dx} = \frac{3u}{u^2 - 1}$$

$$x\frac{du}{dx} = \frac{3u}{u^2-1} - u$$

$$x\frac{du}{dx} = \frac{4u - u^3}{u^2 - 1}$$

$$\int \frac{u^2-1}{4u-u^3}\,du = \int \frac{1}{x}\,dx \qquad \textbf{1}$$

Now, $\dfrac{u^2-1}{4u-u^3} = \dfrac{u^2-1}{u(2-u)(2+u)}$

$$= \frac{A}{u} + \frac{B}{2-u} + \frac{C}{2+u}$$

$\therefore \quad u^2 - 1 = A(2-u)(2+u) + B(2+u)u + Cu(2-u)$

Putting $2 - u = 0$

$\therefore \quad B = \dfrac{3}{8}$

Put $u = 0$

$\therefore \quad A = -\dfrac{1}{4}$

Again put $2 + u = 0$ **1**

$\therefore \quad C = -\dfrac{3}{8}$

Now from $\displaystyle\int \frac{u^2-1}{u(4-u^2)}\,d\nu = \int \frac{1}{x}\,dx$

$$= -\frac{1}{4}\int \frac{1}{u}du + \frac{3}{8}\int \frac{1}{(2-u)}du - \frac{3}{8}\int \frac{1}{2-u}du$$

$$= \int \frac{1}{x}\,dx = -\frac{1}{4}\log|u| - \frac{3}{8}|u-2| - \frac{3}{8}$$

$\log|u+2| = \log|x| + \log c$

$$-\frac{1}{4}\log\left|\frac{y}{x}\right| - \frac{3}{8}\log\left|\frac{y}{x} - 2\right| - \frac{3}{8}\log\left|\frac{y}{x} + 2\right|$$

$= \log|x| + \log c$ **1**

19. Let y denote the number of bacteria at any instant t, then according to the problem,

$$\frac{dy}{dt} \propto y \Rightarrow \frac{dy}{y} = k\,dt \qquad ...(1)$$

where k is the constant of proportionality, taken to be positive.

On integrating (1), we get

$$\log y = kt + C \qquad ...(2)$$

where C is a parameter.

Let y_0 be the initial number of bacteria *i.e.*, at $t = 0$.

Using this in (2), we get

$$C = \log y_0$$

So, we have $\log y = kt + \log y_0$

$$\Rightarrow \quad \log \frac{y}{y_0} = kt \qquad ...(3)$$

According to the problem,

$$y = \left(y_0 + \frac{10}{100} y_0\right)$$

$$= \frac{11y_0}{10}, \text{ when } t = 2 \quad ½$$

So, from (3), we get

$$\log \frac{\frac{11y_0}{10}}{y_0} = k(2)$$

$$\Rightarrow \quad k = \frac{1}{2} \log \frac{11}{10} \qquad ...(4)$$

Using (4) in (3), we get

$$\log \frac{y}{y_0} = \frac{1}{2}\left(\log \frac{11}{10}\right) t \quad ...(5) \quad ½$$

Let the number of bacteria become 1,00,000 to 2,00,000 in t_1 hours *i.e.,* $y = 2y_0$, when $t = t_1$ hours.

Using (5), we get

$$\log \frac{2y_0}{y_0} = \frac{1}{2}\left(\log \frac{11}{10}\right) t_1$$

$$\therefore \quad t_1 = \frac{2 \log 2}{\log \frac{11}{10}}$$

Hence, the required numbers of hours

$$= \frac{2 \log 2}{\log \frac{11}{10}} \qquad \mathbf{1}$$

(i) Diffrential equation ½

(ii) Bacteria is harmful to health. ½

20. Let $\vec{a} = \frac{1}{7}(2\hat{i} + 3\hat{j} + 6\hat{k})$

$$\vec{b} = \frac{1}{7}(3\hat{i} - 6\hat{j} + 2\hat{k})$$

and $\vec{c} = \frac{1}{7}(6\hat{i} + 2\hat{j} - 3\hat{k})$

Now, $|\vec{a}| = \sqrt{\left(\frac{2}{7}\right)^2 + \left(\frac{3}{7}\right)^2 + \left(\frac{6}{7}\right)^2}$ **1**

$$= \frac{1}{7}\sqrt{4 + 9 + 36}$$

$$= \frac{1}{7}\sqrt{49} = \frac{1}{7} \times 7 = 1$$

$$|\vec{b}| = \sqrt{\left(\frac{3}{7}\right)^2 + \left(\frac{-6}{7}\right)^2 + \left(\frac{2}{7}\right)^2}$$

$$= \frac{1}{7}\sqrt{9 + 36 + 4}$$

$$= \frac{1}{7}\sqrt{49} = \frac{1}{7} \times 7 = 1$$

$$|\vec{c}| = \sqrt{\left(\frac{6}{7}\right)^2 + \left(\frac{2}{7}\right)^2 + \left(\frac{-3}{7}\right)^2}$$

$$= \frac{1}{7}\sqrt{36 + 4 + 9}$$

$$= \frac{1}{7}\sqrt{49} = \frac{1}{7} \times 7 = 1$$

$\therefore \vec{a}, \vec{b}, \vec{c}$ are unit vectors.

Now, $\vec{a} \cdot \vec{b} = \frac{1}{49}[2(3) + (3)(-6) + (6)(2)]$

$$= \frac{1}{49}(6 - 18 + 12) = 0 \qquad \mathbf{1}$$

$$\Rightarrow \quad \vec{a} \perp \vec{b}$$

$$\vec{b} \cdot \vec{c} = \frac{1}{49}[(3)(6) + (-6)(2) + (2)(-3)]$$

$$= \frac{1}{49}(18 - 12 - 6) = 0$$

$$\Rightarrow \quad \vec{b} \perp \vec{c}$$

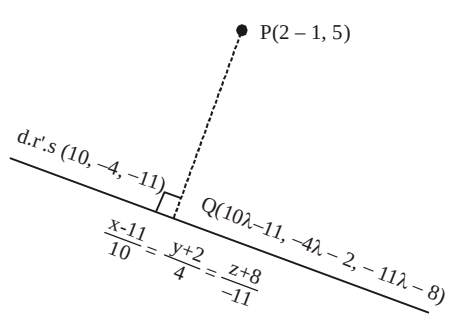
and $\vec{c} \cdot \vec{a} = \frac{1}{49}[(6)(2) + (2)(3) + (-3))6)]$

$$= \frac{1}{49}[12 + 6 - 18] = 0$$

$$\Rightarrow \quad \vec{c} \perp \vec{a}$$

Hence $\vec{a}$, $\vec{b}$ and $\vec{c}$ are three mutually perpendicular unit vectors.

21.



Equation of given line is,

$$\frac{x-11}{10} = \frac{y-2}{-4} = \frac{z-8}{-11}$$

$\therefore$ Direction ratio's of lines are l, m and n = 10, – 4 and – 11 respectively.

Let, $\frac{x-11}{10} = \frac{y+2}{-4} = \frac{z+8}{-11} = \lambda$ **1**

$\therefore$ Coordinate of general point Q is given by line are $x = 10\lambda + 11$, $y = 4\lambda - 2$, $z = -11\lambda - 8$

$\therefore$ Direction ratio's of the line joining this point Q to P are

$10\lambda + 11 - 2, 4\lambda - 2 - (-1), -(11\lambda) - 8 - 5$ or $10\lambda + 9, -4\lambda - 1, -11\lambda - 13$.

If PQ is perpendicular to given line, then

$l_1l_2 + m_1m_2 + n_1n_2 = 0$ **1**

$\Rightarrow 10(10\lambda + 9) - 4(-4\lambda - 1) - 11(-11\lambda - 13) = 0$

$\Rightarrow 100\lambda + 90 + 16\lambda + 4 + 121\lambda + 143 = 0$

$\Rightarrow 237\lambda + 237 = 0$

$\Rightarrow 273\lambda = -273$

$\Rightarrow \lambda = -1$ **1**

$\therefore$ Co-ordinate of foot of $\perp$ Q are

$x = 10(-1) + 11 = 1$

$y = -4(-1) - 2 = 2$

$z = -11(-1) - 8 = 3$

$\therefore$ Q are (1, 2, 3) and $\perp$ distance of P from the line = PQ

$= \sqrt{(1+2)^2 + (2+1)^2 + (3-5)^3}$

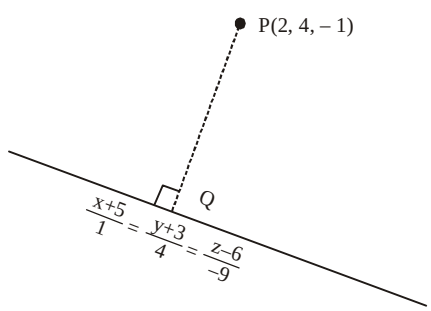
$= \sqrt{1+9+4}$

$= \sqrt{14}$ unit. **1**

OR

Equation of given line is

$$\frac{x+5}{1} = \frac{y+3}{4} = \frac{z+6}{-9} \quad ...(i)$$



and given point P is (2, 4, – 1)

Let the general point on the line is Q.

$\because \frac{x+5}{1} = \frac{y+3}{4} = \frac{z+6}{-9} = r$,(say) **1**

Hence, coordinates of Q are

$x = r - 5$,

$y = 4r - 3$,

$z = -9r + 6$

i.e., Q is $(r - 5, 4r - 3, -9r + 6)$

Direction ratios of PQ are

$r - 5 - 2, 4r - 3 - 4, -9r + 6 - (-1)$ **1**

or, $r - 7, 4r - 7, -9r + 7$

$\because$ PQ is perpendicular to the line (i)

$\therefore l_1l_2 + m_1m_2 + n_1n_2 = 0$

$\Rightarrow 1.(r - 7) + 4(4r - 7) - 9(-9r + 7) = 0$

$r - 5 - 2, 4r - 3 - 4, -9r + 6 - (-1)$

or, $r - 7, 4r - 7, -9r + 7$

$\because$ PQ is $\perp$ to the line.

$\therefore l_1l_2 + m_1m_2 + n_1n_2 = 0$

$\Rightarrow 1(r - 7) + 4(4r - 7) - 9(-9r + 7) = 0$

$\Rightarrow r - 7 + 16r - 28 + 81r - 63 = 0$

$\Rightarrow 98r - 98 = 0$

$\Rightarrow 98r = 98$

$\Rightarrow r = 1$ **1**

Putting $r = 1$ in coordinates of Q.

$x = 1 - 7 = -6$

$y = 4 \times 1 - 7 = -3$

$z = -9 \times 1 + 7 = -2$

$\therefore$ Q is (–6, – 3, – 2)

$\therefore$ PQ $= \sqrt{(2+6)^2 + (4+3)^2 + (-1+2)^2}$

$= \sqrt{8^2 + 7^2 + 1^2}$

$= \sqrt{64 + 49 + 1}$

PQ $= \sqrt{114}$ **1**

22. Let E_1, E_2 and A be the events defined as follows :

E_1 = 5 occurs

E_2 = 5 does not occur

A = a report that it is a 5.

We have,

$P(E_1) = \frac{1}{6}$

and $P(E_2) = 1 - \frac{1}{6} = \frac{5}{6}$ **1**

Now, $P\left(\frac{A}{E_1}\right)$ = Probability that A speaks truth

$= \frac{8}{10} = \frac{4}{5}$

and $P\left(\frac{A}{E_2}\right)$ = Probability that A does not speak truth

$= 1 - \frac{4}{5} = \frac{1}{5}$. **1**

We have to find $P\left(\frac{E_1}{A}\right)$ *i.e.,* the probability that there is 5 on the dice given that A has reported that there is 5. By Baye's theorem, we have

$$P\left(\frac{E_1}{A}\right) = \frac{P(E_1) \times P\left(\frac{A}{E_1}\right)}{P(E_1) \times P\left(\frac{A}{E_1}\right) + P(E_2) \times P\left(\frac{A}{E_2}\right)} \quad \textbf{1}$$

$$= \frac{\frac{1}{6} \times \frac{4}{5}}{\frac{1}{6} \times \frac{4}{5} + \frac{5}{6} \times \frac{1}{5}} = \frac{4}{5} \quad \textbf{1}$$

Section–C

23. Let the cost of onion, wheat and rice per kg be Rs. x, Rs. y, Rs. z respectively.

The cost of 4 kg onion, 3 kg wheat and 2 kg rice

$= \text{Rs. } 60$

$\therefore \quad 4x + 3y + 2z = 60 \quad ...(1)$

The cost of 2 kg onion, 4 kg wheat and 6 kg rice

$= \text{Rs. } 90$

$\therefore \quad 2x + 4y + 6z = 90 \quad ...(2)$

The cost of 6 kg onion, 2 kg wheat and 3 kg rice

$= \text{Rs. } 70$

$6x + 2y + 3z = 70 \quad ...(3)$ **1**

$\therefore$ The system of equation is

$4x + 3y + 2z = 60$

Dividing eq. (2) by 2, we get

$x + 2y + 3z = 45$

$6x + 2y + 3z = 70$

which may be written as

$AX = B$ or $X = A^{-1} B$

Where, $A = \begin{bmatrix} 4 & 3 & 2 \\ 1 & 2 & 3 \\ 6 & 2 & 3 \end{bmatrix}$,

$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 60 \\ 45 \\ 70 \end{bmatrix}$

$A_{11} = (-1)^{1+1} \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix} = 0$ **1**

$A_{12} = (-1)^{1+2} \begin{bmatrix} 1 & 3 \\ 6 & 3 \end{bmatrix} = -(3 - 18) = 15$

$A_{13} = (-1)^{1+3} \begin{bmatrix} 1 & 2 \\ 6 & 2 \end{bmatrix} = 2 - 12 = 10$

$A_{21} = (-1)^{2+1} \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} = -(9 - 4) = -5$

$A_{22} = (-1)^{2+2} \begin{bmatrix} 4 & 2 \\ 6 & 3 \end{bmatrix} = 12 - 12 = 0$

$A_{23} = (-1)^{2+3} \begin{bmatrix} 4 & 3 \\ 6 & 2 \end{bmatrix} = -(8 - 18) = 10$

$A_{31} = (-1)^{3+1} \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} = 9 - 4 = 5$

$A_{32} = (-1)^{3+2} \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix} = -(12 - 2) = -10$

$A_{33} = (-1)^{3+3} \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix} = 8 - 3 = 5$

$$\text{adj } A = \begin{bmatrix} 0 & 15 & -10 \\ -5 & 0 & 10 \\ 5 & -10 & 5 \end{bmatrix}^T$$

$$= \begin{bmatrix} 0 & -5 & 5 \\ 15 & 0 & -10 \\ -10 & -10 & 5 \end{bmatrix} \quad \textbf{1}$$

$|A| = 4 \times 0 + 3 \times 15 + 2 \times (-10)$

$= 45 - 20 = 25$

$$A^{-1} = \frac{1}{|A|} \text{adj } A = \frac{1}{25} \begin{bmatrix} 0 & -5 & 5 \\ 15 & 0 & -10 \\ 10 & 10 & 5 \end{bmatrix}$$

$$= \frac{1}{5} \begin{bmatrix} 0 & -1 & 1 \\ 3 & 0 & -2 \\ -2 & 2 & 1 \end{bmatrix}$$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1} B = \frac{1}{5} \begin{bmatrix} 0 & -1 & 1 \\ 3 & 0 & -2 \\ -2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 60 \\ 45 \\ 70 \end{bmatrix} \quad \textbf{1}$$

$$= \frac{1}{5} \begin{bmatrix} 0 \times 60 - 1 \times 45 + 1 \times 70 \\ 3 \times 60 + 0 \times 45 - 2 \times 70 \\ -120 + 2 \times 45 + 1 \times 70 \end{bmatrix}$$

$$= \frac{1}{5} \begin{bmatrix} -45 + 70 \\ 180 + 70 \\ -120 + 90 + 70 \end{bmatrix}$$

$$= \frac{1}{5} \begin{bmatrix} 25 \\ 40 \\ 40 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \\ 8 \end{bmatrix}$$

$\therefore \quad x = 5, y = 8, z = 8$

$\Rightarrow$ The cost of onion, wheat and rice per kg are Rs. 5, Rs. 8, Rs. 8 respectively. **1**

Or

The given system of equations can be written as

$$AX = B$$

Where $\quad A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

and $\quad B = \begin{bmatrix} 7 \\ 11 \\ 1 \end{bmatrix}$ **1**

Now, $\quad |A| = 1(0 + 9) - 2(0 - 6) + 1(-3 - 0)$

$= 9 + 12 - 3 = 18 \neq 0$ **1**

$\therefore$ A^{-1} exists and it is given by

$$A^{-1} = \frac{1}{|A|} \text{ adj } A$$

$$= \frac{1}{18} \begin{bmatrix} 9 & -3 & 6 \\ 6 & -2 & -2 \\ -3 & 7 & -2 \end{bmatrix}$$ **1**

By equation (1), we get

$$X = A^{-1} B$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{18} \begin{bmatrix} 9 & -3 & 6 \\ 6 & -2 & -2 \\ -3 & 7 & -2 \end{bmatrix} \begin{bmatrix} 7 \\ 11 \\ 1 \end{bmatrix}$$ **1**

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{18} \begin{bmatrix} 36 \\ 18 \\ 54 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$ **1**

Thus, $\quad x = 2, y = 1, z = 3.$ **1**

Note : Students are advised to first check A^{-1} exists or not. For checking this, they simply need to verify that

$$|A| \neq 0$$ **1**

24. Let the line through P(3, 4) meets x-axis in A(a, 0) then equation of line PA

$$\Rightarrow \quad (y - 0) = \frac{1 - 0}{3 - a}(x - a)$$

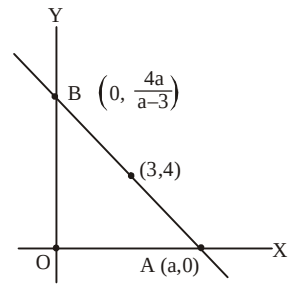
$$\Rightarrow \quad (3 - a)y = 4(x - a) \quad \text{...(1)}$$

Line (1) meets y-axis when $x = 0$

$$\therefore \quad (3 - a)y = 4(0 - a)$$ **1**

$$\Rightarrow \quad y = \frac{4a}{a - 3}$$ **1**

So, line (1) meets y-axis in B $\left(0, \frac{4a}{a-3}\right)$



Area of triangle, OAB

$$= \frac{1}{2}(OA)(OB)$$ **1**

$$\Delta = \frac{1}{2}(a)\left(\frac{4a}{a-3}\right)$$

$$\Delta = \frac{2a^2}{a-3}$$

$$\frac{d\Delta}{da} = \frac{(a-3)4a - 2a^2}{(a-3)^2}$$ **1**

$$= \frac{2a^2 - 12a}{(a-3)^2}$$

Now $\quad \frac{d\Delta}{da} = 0$

$\Rightarrow \quad 2a^2 - 12a = 0$

$\Rightarrow \quad a = 6$

$$\frac{d^2\Delta}{da^2} = \frac{(a-3)^2(4a-12) - (2a^2 - 12a)(a-3)2}{(a-3)^4}$$ **1**

$$= \frac{36}{(a-3)^3}$$

At $a = 6$, $\quad \frac{d^2\Delta}{da^2} = \frac{36}{27} = \frac{4}{3} > 0$

$\therefore$ Δ is minimum at $a = 6$

Equation of line is $(3 - 6)y = 4(x - 6)$ **1**

or $\quad 4x + 2y = 24$

and minimum area of triangle

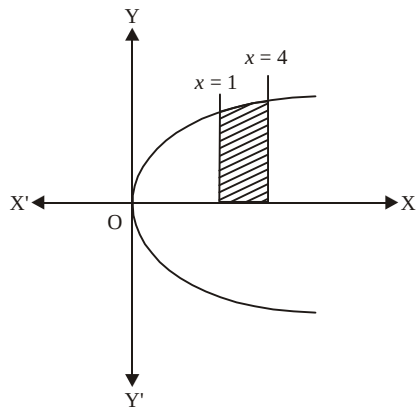
$$= \frac{2(6)^2}{6-3} = \frac{2 \times 36}{3}$$

$= 24$ sq. units. **1**

25. The required area

= Area of the shaded portion

$$= \int_1^4 y\,dx = \int_1^4 \sqrt{4x}\,dx$$ **1**



2

$$= 2.\left[\frac{x^{3/2}}{3/2}\right]_1^4 = \frac{4}{3}\,[4^{3/2} - 1] \qquad \textbf{2}$$

$$= \frac{28}{3} \text{ sq. units.} \quad \textbf{1}$$

OR

Here $f(x) = x^2 + x + 1$,

$a = 0, b = 2$

$nh = b - a = 2$

By the formula,

$$\int_a^b f(x)dx = \lim_{h\to 0} h\,[f(a) + f(a+h) + \ldots + f(a + \overline{n-1}\,h)], \quad \text{(where } nh = b - a.)$$

$$\Rightarrow \int_0^2 (x^2 + x + 1)\,dx \qquad \textbf{1}$$

$$= \lim_{h\to 0} h\,[(0 + 0 + 1) + (h^2 + h + 1) + (2^2h^2 + 2h + 1) + \ldots + (n-1)^2h^2 + (n-1)h + 1], \quad \text{(where } nh = 2)$$

$$= \lim_{h\to 0} h\,[\{1^2 + 2^2 + \ldots + (n-1)^2\}\,h^2 + (1 + 2 + \ldots + n + 1)\,h + n], \quad \text{where } nh = 2 \qquad \textbf{1}$$

$$= \lim_{h\to 0} h\left[\frac{1}{6}(n-1)n(2n-1)h^2 + \frac{(n-1)n}{2}h + n\right] \qquad \textbf{1}$$

$$= \lim_{h\to 0}\left[\frac{1}{6}(nh-h)nh(2nh-h) + \frac{1}{2}(nh-h)nh + nh\right]$$

$$= \frac{1}{6}(2-0)\times 2(2\times 2 - 0) + \frac{1}{2}(2-0)\times 2 + 2$$

$$= \frac{8}{3} + 2 + 2 = \frac{20}{3} \qquad \textbf{1}$$

26. $$\int_0^{\pi} \frac{dx}{5 + 4\cos x}$$

$$= \int_0^{\pi} \frac{dx}{5\left(\cos^2\frac{x}{2} + \sin^2\frac{x}{2}\right) + 4\left(\cos^2\frac{x}{2} - \sin^2\frac{x}{2}\right)} \qquad \textbf{1}$$

$$= \int_0^{\pi} \frac{dx}{9\cos^2\frac{x}{2} + \sin^2\frac{x}{2}}$$

Dividing by $\cos^2\frac{x}{2}$ numerator and denominator

$$= \int_0^{\pi} \frac{\sec^2\frac{x}{2}}{9 + \tan^2\frac{x}{2}}\,dx \qquad \textbf{1}$$

Now, let $\tan\frac{x}{2} = t \Rightarrow \frac{1}{2}\sec^2\frac{x}{2}\,dx = dt$

Also, when $x = 0$, then $t = \tan 0 = 0$ and when $x = \pi$, then $t = \tan\frac{\pi}{2} = \infty$ 1

$$\therefore \quad I = 2\int_0^{\infty} \frac{dt}{3^2 + t^2} \qquad \textbf{1}$$

$$= \frac{2}{3}\left[\tan^{-1}\frac{t}{3}\right]_0^{\infty} \qquad \textbf{1}$$

$$= \frac{2}{3}[\tan^{-1}\infty - \tan^{-1}0] \quad \textbf{1}$$

$$= \frac{2}{3}\left(\frac{\pi}{2} - 2\right) = \frac{\pi}{3} \qquad \textbf{1}$$

27. First two points A and B are, (0, 0, 0) and (2, 1, 1) respectively. **1**

$\therefore$ Direction ratios of AB are 2, 1, 1.

Direction ratios of CD joining the points C(3, 5, – 1), D(4, 3, – 1) are (1, – 2, 0).

If AB $\perp$ CD, then $a_1.a_2 + b_1.b_3 + c_1.c_2 = 0$

i.e., $2.1 + 1.(-2) + 1.0 = 2 - 2 + 0 = 0$

which is true **2**

$\Rightarrow$ AB is perpendicular to CD. **1**

28. Let x kg of fertiliser F_1 and y kg of fertiliser F_2 is be required, we are given that :

Fertiliser	Nitrogen	Phosphoric Acid	Quantity	Cost
F_1	10%	6%	x kg	Rs. 6
F_2	5%	10%	y kg	Rs. 5
Total	14 kg	14 kg		

Quantity of nitrogen in F_1 and F_2

10% of x + 5% of y = $\frac{x}{10} + \frac{x}{20}$

Least quantity of nitrogen = 14 kg **1**

$\Rightarrow \frac{x}{10} + \frac{y}{20} \geq 14$ or $2x + y \geq 280$

Quantity phosphoric acid in fertilisers F_1 and F_2

= 6% of x + 10% of y

$= \frac{6x}{100} + \frac{y}{10}$

Least quantity of phosphoric acid = 14 kg

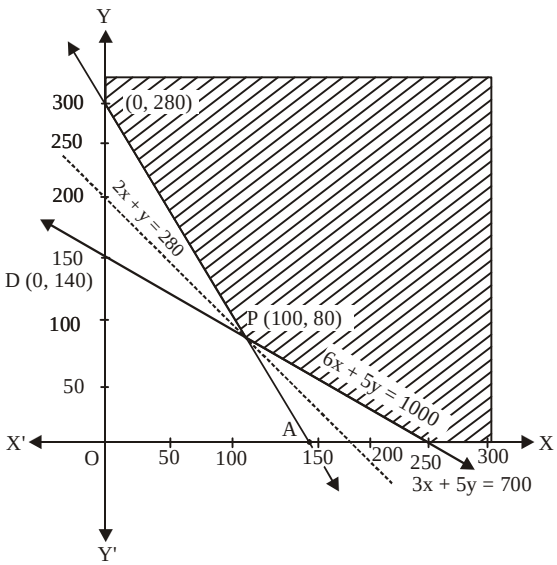
$\Rightarrow \frac{6x}{100} + \frac{y}{10} \geq 14$

or $6x + 10y \geq 1400$ or $3x + 5y \geq 700$

$\therefore$ We have to minimize the cost $Z = 6x + 5y$ and constraints are : $2x + y \geq 280$, $3x + 5y \geq 700$; $x, y \geq 0$.

(i) The line $2x + y = 280$ passes through A (140, 0), B(0, 280). Putting $x = 0$, $y = 0$, in $2x + y \geq 280$, $0 \geq 280$ which is not true. **1**

$\Rightarrow 2x + y \geq 280$ lies on and above AB.



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(ii) The line $3x + 5y = 700$ passes through C $\left(\frac{700}{3}, 0\right)$, D(0, 140). Putting $x = 0$, $y = 0$, is $3x + 5y \geq 700$, $0 \geq 70$ which in not true.

$\Rightarrow 3x + 5y \geq 700$ lies on and above CD.

(iii) $x \geq 0$ lies on and to the right of y-axis.

(iv) $y \geq 0$ lies on and above the x-axis. **1**

The shaded area YBPCX is the feasible region where P is the intersection of AB and CD *i.e.,*

$$2x + y = 280 \quad ...(1)$$
$$3x + 5y = 700 \quad ...(2)$$

Multiplying (1) by 5 and subtracting (2) from it $7x = 700$.

$\therefore \quad x = 100$

From (1), $\quad y = 80$

$\therefore$ The point P is (100, 80).

The value of Z at B, P are

At B (0, 280), $\quad Z = 6x + 5y = 0 + 5 \times 280$
$= 1400$

At P(100, 80), $\quad Z = 6 \times 100 + 5 \times 80$
$= 600 + 400 = 1000$

At C $\left(\frac{700}{3}, 0\right)$, $\quad Z = 6 \times \frac{700}{3} + 0 = 1400$

$\therefore$ Z is minimum at P(100, 80) where Z = 1000

But, the feasible region is unbounded.

Consider the inequality $6x + 5y < 1000$. **1**

$6x + 5y = 1000$ passes through $\left(\frac{500}{3}, 0\right)$ and (0, 200).

Putting $x = 0$, $y = 0$ in $6x + 5y < 1000$, $0 < 1000$ which is true.

$\therefore$ $6x + 5y < 1000$ lies below the line $6x + 5y = 1000$.

There is no point in common between feasible region and $6x + 5y < 1000$.

Hence, minimum cost of Z is Rs. 1000 when quantities of fertiliser F_1 and F_2 are 100 kg and 80 kg respectively. **1**

29. Events A fails and B fails are denoted by $\overline{A}$ and $\overline{B}$ respectively.

We have $\quad P(\overline{A}) = 0.2$

and P(A and B fail) $= 0.15$

i.e., $\quad P(\overline{A} \cap \overline{B}) = 0.15$

$P(\overline{B}$ alone$) = P(\overline{B}) - P(\overline{A} \cap \overline{B})$

$= 0.15$ 1

Now, $\quad 0.15 = P(\overline{B}) - 0.15$

$\therefore \quad P(\overline{B}) = 0.30$

(i) $\quad P(\overline{A}/\overline{B}) = \frac{P(\overline{A} \cap \overline{B})}{P(\overline{B})}$

$= \frac{0.15}{0.30} = \frac{1}{2} = 0.5$ 1

(ii) P(A fails alone) $= P(\overline{A}$ alone)

$= P(\overline{A}) - P(\overline{A} \cap \overline{B})$ 1

$= 0.20 - 0.15 = 0.05.$ 1

●●

Sample Question Paper—8

Section–A

1. Relation R in the set A = {1, 2, 3, 4, 5, 6} as R = {(x, y) : y is divisible by x}

(a) Putting $y = x$, x is divisible by $x \Rightarrow$ R is reflexive.

(b) If y is divisible by x, then x in not divisible by y when $x \neq y \Rightarrow$ R is not symmetric. **½**

(c) If y is divisible by x and z is divisible by y, then z is divisible by x *e.g.*, 2 is divisible by 1, 4 is divisible by 2. **½**

2. $\tan^{-1}(1) + \cos^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\left(-\frac{1}{2}\right)$

Now $\tan^{-1} 1 = \frac{\pi}{4}$

$\because$ Range of principle value branch of $\cos^{-1}$ is $[0, \pi]$

$$\cos^{-1}\left(-\frac{1}{2}\right) = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$$

and $\sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$ **½**

since range of principle value branch of $\sin^{-1}$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$$\therefore \tan^{-1}(1) + \cos^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\left(-\frac{1}{2}\right)$$

$$= \frac{\pi}{4} + \frac{2\pi}{3} - \frac{\pi}{6}$$

$$= \frac{3\pi + 8\pi - 2\pi}{12} = \frac{9\pi}{12} = \frac{3\pi}{4}$$ **½**

3. $A = \begin{bmatrix} 3y+7 & 5 \\ y+1 & 2-3x \end{bmatrix}$, $B = \begin{bmatrix} 0 & y-2 \\ 8 & 4 \end{bmatrix}$

A = B, if $3x + 7 = 0$

$\therefore$ $x = -\frac{7}{3}$ **½**

$5 = y - 2$ or $y = 7$

$y + 1 = 8.$

$\therefore$ $y = 7$

$2 - 3x = 4,$

$x = -\frac{2}{3}$

Here x has two values viz $-\frac{7}{3}, -\frac{2}{3}$ **½**

$\therefore$ Not possible to find values of x and y.

4. There are 9 entries in a 3 × 3 matrix. Each place can be filled with 0 to 1.

$\therefore$ 9 places can be filled in $2^9 = 512$ ways **½**

Number of such matrices = 512 **½**

5. (i) $\begin{vmatrix} 2 & 4 \\ 5 & 1 \end{vmatrix} = \begin{vmatrix} 2x & 4 \\ 6 & x \end{vmatrix}$ or $2 - 20 = (2x^2 - 24)$

or $-18 = 2x^2 - 24$ or $2x^2 = 6$, $x = \pm\sqrt{3}$ **½**

(ii) $\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = \begin{vmatrix} x & 3 \\ 2x & 5 \end{vmatrix}$

or $2 \times 5 - 4 \times 3 = 5 \times x - 2x \times 3$

or $10 - 12 = 5x - 6x \Rightarrow -2 = -x$

$\therefore$ $x = 2.$ **½**

6. $\frac{d}{dx}(ax+b)^3 = 3a(ax+b)^2$

$$(ax+b)^2 = \frac{1}{3a}\frac{d}{dx}(ax+b)^3$$

$$= \frac{d}{dx}\left(\frac{1}{3a}(ax+b)^3\right)$$ **½**

$$(ax+b)^2 = \frac{d}{dx}\left(\frac{1}{3a}(ax+b)^3\right)$$

$\therefore$ An anti-derivative of $(ax + b)^2$ is $\frac{1}{3a}(ax + b)^3 + c$. **½**

7. $\int (\sin 2x - 4e^{3x})\, dx$

$= \int \sin 2x\, dx - \int 4e^{3x} dx$ **½**

$= \left(-\frac{\cos 2x}{2}\right) - 4\int e^{3x} dx$

$= \left(-\frac{\cos 2x}{2}\right) - \frac{4}{3}e^{3x} + c$ **½**

8. $\vec{a} \times \vec{b} = \vec{c} \times \vec{d}$, $\vec{a} \times \vec{c} = \vec{b} \times \vec{d}$

Consider $(\vec{a} - \vec{d}) \times (\vec{b} - \vec{c})$

$= \vec{a} \times \vec{b} - \vec{a} \times \vec{c} - \vec{d} \times \vec{b} + \vec{d} \times \vec{c}$ **½**

$= (\vec{c} \times \vec{d}) - (\vec{b} \times \vec{d}) + (\vec{b} \times \vec{d}) - (\vec{c} \times \vec{d})$

$= 0$

$\Rightarrow$ $\vec{a} - \vec{d}$ is parallel to $\vec{b} - \vec{c}$. **½**

9. Unit vector $= \dfrac{3\hat{i} - 2\hat{j} + 6\hat{k}}{\sqrt{(3)^2 + (-2)^2 + (6)^2}}$

$= \dfrac{3\hat{i} - 2\hat{j} + 6\hat{k}}{\sqrt{9 + 4 + 36}}$ **½**

$= \dfrac{1}{7}(3\hat{i} - 2\hat{j} + 6\hat{k})$ **½**

10. For two vectors to be perpendicular

$\vec{a} \cdot \vec{b} = 0$

$(2\hat{i} + \lambda\hat{j} + \hat{k}).(\hat{i} - 2\hat{j} + 3\hat{k}) = 0$

$(2)(1) + \lambda(-2) + 3(1) = 0$

$2 - 2\lambda + 3 = 0$

$5 = 2\lambda$

$\lambda = \dfrac{5}{2}$ **½**

Section–B

11. (i) $f : R \to R$ defined by $f(x) = 3 - 4x$

$f(x_1) = 3 - 3x_1$, $f(x_2) = 3 - 4x_2$

(a) $f(x_1) = f(x_2)$

$\Rightarrow$ $3 - 4x_1 = 3 - 4x_1 - 3 = -4x_2$

$\therefore$ $x_1 = x_2$ **1**

This shows that f is one-one.

(b) $f(x) = y = 3 - 4x$

$\therefore$ $x = \dfrac{3 - y}{4}$

For every value of y belonging to its co-domain, there is a pre-image in its domain.

$\Rightarrow$ f is onto.

Hence, f is one-one and onto. **1**

(ii) $f : R \to R$ is given by

$f(x) = 1 + x^2$

(a) $f(1) = 1 + 1 = 2$, $f(-1) = 1 + 1 = 2$

$\therefore$ $f(-1) = f(1) = 2$

i.e., -1 and 1 have the same image 2. **1**

$\Rightarrow$ f is not one-one.

(b) No negative number belonging to its co-domain has its pre-image in its domain.

$\Rightarrow$ f in not onto.

Thus f is neither one-one nor onto. **1**

12. $\tan^{-1}(2x) + \tan^{-1}(3x) = \dfrac{\pi}{4}$

$\tan^{-1}\left(\dfrac{2x + 3x}{1 - 6x^2}\right) = \dfrac{\pi}{4}$ **1**

$\dfrac{5x}{1 - 6x^2} = 1$

$1 - 6x^2 = 5x$

$6x^2 + 5x - 1 = 0$

$6x^2 + 6x - x - 1 = 0$

$6x(x + 1) - 1(x + 1) = 0$

$(6x - 1)(x + 1) = 0$

$x = \dfrac{1}{6}, x = -1$

$x = \dfrac{1}{6}$ **1**

13. Let $\Delta = \begin{vmatrix} y+z & x & y \\ z+x & z & x \\ x+y & y & z \end{vmatrix}$ **1**

Operating $R_1 \to R_1 + (R_2 + R_3)$, we get

$$\Delta = \begin{vmatrix} 2(x+y+z) & x+y+z & x+y+z \\ z+x & z & x \\ x+y & y & z \end{vmatrix}$$

$$\Rightarrow \Delta = (x + y + z)\begin{vmatrix} 2 & 1 & 1 \\ z+x & z & x \\ x+y & y & z \end{vmatrix}$$ **1**

Operating $C_1 \to C_1 - (C_2 + C_3)$; $C_2 \to C_2 - C_3$, we get

$$\Delta = (x + y + z)\begin{vmatrix} 0 & 0 & 1 \\ 0 & z-x & x \\ x-z & y-z & z \end{vmatrix}$$ **1**

$= (x + y + z)[1\{0 - (z - x)(x - z)\}]$

$= (x + y + z)(x - z)^2$. **1**

OR

Let $\Delta = \begin{vmatrix} x+1 & x+2 & x+a \\ x+2 & x+3 & x+b \\ x+3 & x+4 & x+c \end{vmatrix}$

Applying $R_3 \to R_3 - R_2$ and $R_2 \to R_2 - R_1$, we get **1**

$$\Delta = \begin{vmatrix} x+1 & x+2 & x+a \\ 1 & 1 & b-a \\ 1 & 1 & c-b \end{vmatrix} \quad ...(1)$$ **1**

Since a, b, c are in A.P., then ...(2)

$$b - a = c - b \quad \textbf{1}$$

Using (2) in (1), we get

$$R_2 = R_3$$

$$\therefore \quad \Delta = 0. \quad \textbf{1}$$

14. Let $y = \cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right)$

Now $1 + \sin x = \cot^2\frac{x}{2} + \sin^2\frac{x}{2} + 2\cos\frac{x}{2}\sin\frac{x}{2}$

$$= \left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2 \quad \textbf{1}$$

$$\left(\because \text{ for } \frac{x}{2} \in \left(0, \frac{\pi}{4}\right)\right)$$

$$\cos\frac{x}{2} > \sin\frac{x}{2}$$

Similarly $1 - \sin x = \left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2$

$$\therefore \quad \cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right) \quad \textbf{1}$$

$$= \cot^{-1}\left(\frac{\sqrt{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}+\sqrt{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}}{\sqrt{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}-\sqrt{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}}\right)$$

$$= \cot^{-1}\left(\frac{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)+\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)}{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)-\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)}\right) \quad \textbf{1}$$

$$= \cot^{-1}\left(\frac{2\cos\frac{x}{2}}{2\sin\frac{x}{2}}\right) = \cot^{-1}\left(\cot\frac{x}{2}\right) = \frac{x}{2}$$

Hence $\frac{dy}{dx} = \frac{1}{2}$

15. (i) Let v is the velocity of the car at t sec.

Now $x = t^2\left(2 - \frac{t}{3}\right)$

Therefore, $v = \frac{dx}{dt} = 4t - t^2$

$$= t(4 - t)$$

Thus, $v = 0$ gives $t = 0$ and/or $t = 4$ **1**

Now $v = 0$ at P as well as at Q and at P, $t = 0$. So, at Q, $t = 4$. Thus, the car will reach the point Q after 4 sec. Also the distance travelled in 4 seconds is given by

$$x|_{t=4} = 4^2\left(2 - \frac{4}{3}\right) \quad \textbf{1}$$

$$= 16\left(\frac{2}{3}\right) \quad \textbf{1}$$

16. Let $y = u.v.w = u.(vw)$

Differentiating both sides

(i) $\frac{dy}{dx} = u'.(vw) + u\frac{d}{dx}(vw)$

$= u'.(vw) + u[v'w + vw']$ **1**

$= u'.v.w + uv'w + uvw'$

$= \frac{du}{dx}.v.w + u.\frac{dv}{dx}.w + u.v.\frac{dw}{dx}$

(ii) $y = u.v.w$

Taking log of both sides

$\log y = \log u + \log v + \text{low } w,$

$[\because \log mn = \log m + \log n]$ **1**

Differentiating both sides

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{u}\frac{du}{dx}+\frac{1}{v}\frac{du}{dx}+\frac{1}{w}\frac{dw}{dx}$$

$$\therefore \quad \frac{dy}{dx} = y\left(\frac{1}{u}\frac{du}{dx}+\frac{1}{v}\frac{du}{dx}+\frac{1}{w}\frac{dw}{dx}\right)$$

$$= uvw\left(\frac{1}{u}\frac{du}{dx}+\frac{1}{v}\frac{du}{dx}+\frac{1}{w}\frac{dw}{dx}\right) \quad \textbf{1}$$

$$= vw\frac{du}{dx} + uw\frac{du}{dx} + uv\frac{dw}{dx}$$

$$= \frac{du}{dx}.v.w + u.\frac{du}{dx}.w + u.v.\frac{dw}{dx} \quad \textbf{1}$$

17. $\frac{3x-2}{(x+1)^2(x+3)} = \frac{A}{(x+1)} + \frac{A}{(x+1)^2} + \frac{C}{(x+3)}$

$\Rightarrow A(x+1)(x+3) + B(x+3) + C(x+1)^2 = 3x - 2$

$\Rightarrow A(x^2 + 4x + 3) + B(x+3) + C(x^2 + 1 + 2x) = 3x - 2$

$\Rightarrow x^2(A + C) + x(4A + B + 2C) + (3A + 3B + C) = 3x - 2$

Comparing the coefficients of both sides, we get

$A + C = 0$ ...(1)

$4A + B + 2C = 3$ ...(2)

$$3A + 3B + C = -2 \quad ...(3)$$

From equation (1), $A = -C$ **1**

Substituting it in equation (2) and equation (3), we get

$$-4C + B + 2C = 3$$

$$\Rightarrow \quad B - C = 3 \quad ...(4)$$

and $-C + 3B + C = -2$

$$\Rightarrow \quad 3B - 2C = -2 \quad ...(5)$$

From equation (4) and equation (5), we get

$$-2B = 5$$

$$B = -\frac{5}{2}$$

Substituting it in equation (4), we get

$$-\frac{5}{2} - 2C = 3$$

$$2C = -\frac{5}{2} - 3 = -\frac{11}{2} \Rightarrow C = -\frac{11}{4}$$ **1**

$$A = -C = \frac{11}{4}$$

$$I = \int \left\{ \frac{11}{4(x+1)} - \frac{5}{2(x+1)^2} - \frac{11}{4(x+3)} \right\} dx$$

$$= \frac{11}{4}\int \frac{dx}{(x+1)} - \frac{5}{2}\frac{dx}{(x+1)^2} - \frac{11}{4}\int \frac{dx}{(x+3)}$$ **½**

$$= \frac{11}{4}\log(x+1) + \frac{5}{2}.\frac{1}{(x+1)}$$

$$- -\frac{11}{4}\log(x+3) + c.$$ **½**

OR

$$\int \frac{\sin(x-\alpha)}{\sin(x+\alpha)}dx = \int \frac{\sin(x+a-2\alpha)}{\sin(x+\alpha)}dx$$ **1**

$$= \int \frac{\sin(x+\alpha)\cos 2\alpha - \cos(x+\alpha)\sin 2\alpha}{\sin(x+\alpha)}dx$$ **1**

$$= \cos 2\alpha \int dx - \sin 2\alpha \int \cot(x+\alpha)\, dx$$ **1**

$$= x\cos 2\alpha - \sin 2\alpha \log|\sin(x+\alpha)| + c.$$ **1**

18. Consider,

$$y = e^x.\cos x \quad ...(i)$$

$$\Rightarrow \quad y = e^x(-\sin x) + \cos x.\, e^x$$

$$\Rightarrow \quad y' = -e^x \sin x + y \quad ...(ii)$$ **1**

[From (i)]

$$\Rightarrow y'' = -[e^x \cos x + e^x \sin x] + y'$$

$$= -[y + y - y'] + y',$$

[From (i) and (ii)] **1**

$$\Rightarrow \quad y'' - 2y' + 2y = 0$$

As differential equation corresponding to the function $y = e^x \cos x$ is

$$y'' - 2y' + 2y = 0$$

Therefore, $y = e^x \cos x$ is the solution of differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0.$$ **1**

OR

Consider the equation

$(x^2 + 3xy + y^2)\, dx - x^2\, dy = 0$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{x^2 + 3xy + y^2}{x^2}$$

$$= 1 + 3\left(\frac{y}{x}\right) + \left(\frac{y}{x}\right)^2 \quad ...(i)$$

1

Put $y = \nu x$

$$\Rightarrow \quad \frac{dy}{dx} = \nu + x\frac{du}{dx}$$

Substituting in (i), we get

$$\nu + x\frac{d\nu}{dx} = 1 + 3\nu + \nu^2$$

$$\Rightarrow \quad x\frac{d\nu}{dx} = 1 + 2\nu + \nu^2 = (1+\nu)^2$$ **1**

$$\Rightarrow \quad \frac{1}{(\nu+1)^2}d\nu = \frac{dx}{x}$$

Integrating both sides, we get

$$\int \frac{1}{(\nu+1)^2}d\nu = \int \frac{dx}{x}$$

$$\Rightarrow \quad \frac{-1}{\nu+1} = \log|x| + c$$ **1**

$$\Rightarrow \quad \frac{-1}{\frac{y}{x}+1} = \log|x| + c$$

$\Rightarrow \dfrac{-x}{y+x} = \log|x| + c$ is the required solution. **1**

19. Here,

$$2x^2\frac{dy}{dx} - 2xy + y^2 = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2xy - y^2}{2x^2} \quad ...(i)$$ **1**

This is a homogeneous differential equation.

Put $y = \nu x$, so that $\dfrac{dy}{dx} = \nu + x\dfrac{d\nu}{dx}$ in (i), we get

$$v + x\frac{du}{dx} = \frac{2vx^2 - v^2x^2}{2x^2} = v - \frac{v^2}{2}$$

$$\Rightarrow \quad x\frac{dv}{dx} = -\frac{v^2}{2} \Rightarrow 2\frac{dv}{v^2} + \frac{dx}{x} = 0 \qquad \textbf{1}$$

Integrating it, we get

$$-\frac{2}{v} + \log|x| = C$$

or $-\dfrac{2x}{y} + \log|x| = C$, [as $y = vx$]...(ii)

Given $y = e$, when $x = e$, then by (i)

$$-\frac{2e}{e} + \log e = C \Rightarrow -2 + 1 = C \Rightarrow C = -1 \qquad \textbf{1}$$

Put C = – 1 in (i), we get

$$-\frac{2x}{y} + \log(x) = -1 \Rightarrow \frac{2x}{y} = 1 + \log|x|$$

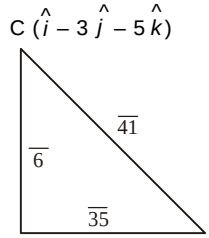
$$\therefore \quad y = \frac{2x}{1 + \log|x|} \qquad \textbf{1}$$

which is the required solution.

20. $\vec{AB} = \vec{b} - \vec{a}$

$$= (2\hat{i} - \hat{j} + \hat{k}) - (3\hat{i} - 4\hat{j} - 4\hat{k})$$

$$= \hat{i} - 3\hat{j} + 5\hat{k} \qquad \textbf{1}$$



$$|\vec{AB}|^2 = (-1)^2 + 3^2 + 5^2$$

$$= 1 + 9 + 25 = 35 \qquad \textbf{1}$$

$$\vec{BC} = \vec{c} - \vec{b}$$

$$= (\hat{i} - 3\hat{j} - 5\hat{k}) - (2\hat{i} - \hat{j} + \hat{k})$$

$$= (1 - 2)\hat{i} + (-3 + 1)\hat{j} + (-5 - 1)\hat{k}$$

$$= -\hat{i} - 2\hat{j} - 6\hat{k}$$

$$\therefore \quad |\vec{BC}|^2 = (-1)^2 + (-2)^2 + (-6)^2$$

$$= 1 + 4 + 36 = 41$$

$$\vec{CA} = \vec{a} - \vec{c} = (3\hat{i} - 4\hat{j} - 4\hat{k}) - (\hat{i} - 3\hat{j} - 5\hat{k})$$

$$= (3 - 1)\hat{i} + (-4 + 3)\hat{j} + (-4 + 5)\hat{k}$$

$$= 2\hat{i} - \hat{j} + \hat{k}$$

$$|\vec{CA}|^2 = 2^2 + (-1)^2 + 1^2 = 4 + 1 + 1 = 6$$

Now, $|\vec{AB}|^2 + |\vec{AC}|^2 = |\vec{BC}|^2$

or $(\vec{AB})^2 + (\vec{AC})^2 = (\vec{BC})^2$

Hence, ΔABC is a right angled triangle. **1**

21. Ist method :

$$\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1} \quad ...(i)$$

and, $$\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1} \quad ....(ii)$$

Let (i) passes through (3, 5, 7) and has directly proportional to 1, – 2, 1.

$\therefore$ Its vector equation is,

$$\vec{r} = \vec{a_1} + \lambda\vec{b_1}$$

$$\vec{a_1} = 3\hat{i} + 5\hat{j} + 7\hat{k}$$

and $$\vec{b_1} = \hat{i} - 2\hat{j} + \hat{k} \qquad \textbf{1}$$

Let line (ii) passes through (–1, – 1, – 1) and directly proportional to 7, – 6, 1.

$\therefore$ Its vector equation is.

$$\vec{r} = \vec{a_2} + \lambda\vec{b_2}$$

$$\vec{a_2} = -\hat{i} - \hat{j} - \hat{k}$$

and $$\vec{b_2} = 7\hat{i} - 6\hat{j} + \hat{k} \qquad \textbf{1}$$

$$\text{S.D.} = \left|\frac{(\vec{a_2} - \vec{a_1}).(\vec{b_2} \times \vec{b_1})}{|\vec{b_1}| \times |\vec{b_2}|}\right|$$

$$(\vec{a_2} - \vec{a_1}) = -\hat{i} - \hat{j} - \hat{k} - (3\hat{i} + 5\hat{j} + 7\hat{k})$$

$$= -\hat{i} - \hat{j} - \hat{k} - 3\hat{i} - 5\hat{j} - 7\hat{k}$$

$$= -4\hat{i} - 6\hat{j} - 8\hat{k}$$

$$\vec{b_1} \times \vec{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 1 \\ 7 & -6 & 1 \end{vmatrix}$$

$$\Rightarrow \hat{i}(-2 + 6) - \hat{j}(1 - 7) + \hat{k}(-6 + 14)$$

$$= 4\hat{i} + 6\hat{j} + 8\hat{k}$$

$$|\vec{b_1} \times \vec{b_2}| = \sqrt{4^2 + 6^2 + 8^2}$$

$$= \sqrt{16 + 36 + 64}$$

$$= \sqrt{116} \qquad \textbf{1}$$

$$(\vec{a_2} - \vec{a_1}).(\vec{b_1} \times \vec{b_2})$$

$$= (-4\hat{i} - 6\hat{j} - 8\hat{k}).(4\hat{i} + 6\hat{j} + 8\hat{k})$$

$= -16 - 36 - 64$

$= -116$

$\therefore$ S.D. $= \left|\dfrac{-116}{\sqrt{116}}\right|$

$= \sqrt{116}$

S.D. $= 2\sqrt{29}$ units. **1**

IInd Method :

Here $\vec{a_1} = 3\hat{i} + 5\hat{j} + 7\hat{i}$, $\vec{b_1} = \hat{i} - 2\hat{j} + \hat{k}$

$\vec{a_2} = -\hat{i} - \hat{j} - \hat{k}$, $\vec{b_2} = 7\hat{i} - 6\hat{j} + \hat{k}$

Shortest distance (S.D.) $= \dfrac{|(\vec{a_2} - \vec{a_1}).(\vec{b_1} \times \vec{b_2})|}{|\vec{b_1} \times \vec{b_2}|}$ **1**

$\vec{a_2} - \vec{a_1} = 4\hat{i} - 6\hat{j} - 8\hat{k}$ **1**

Finding $\vec{b_1} \times \vec{b_2} = 4\hat{i} + 6\hat{j} + 8\hat{k}$

$\therefore$ S.D. $= \dfrac{|-16 - 36 - 64|}{\sqrt{116}}$

$= \dfrac{116}{\sqrt{116}}$

$= \sqrt{116}$

$= 2\sqrt{29}$ units

OR

Ist method :

$$\frac{x+2}{3} = \frac{y+1}{2} = \frac{z-3}{2} = \lambda$$

General points, $(3\lambda - 2, 2\lambda - 1, 2\lambda + 3)$

Its distance from the point A (1, 2, 3) is $3\sqrt{2}$

$\therefore$ $PA^2 = 18$

$(3\lambda - 2 - 1)^2 + (2\lambda - 1 - 2)^2 + (2\lambda + 3 - 3)^2 = 18$

$(3\lambda - 3)^2 + (2\lambda - 3)^2 + (2\lambda)^2 = 18$

$9\lambda^2 + 9 - 18\lambda - 14\lambda^2 + 9 - 12\lambda + 4\lambda^2 = 18$ **1**

$17\lambda^2 - 30\lambda = 18 - 18$

$17\lambda^2 - 30\lambda = 0$

$\lambda(17\lambda - 30) = 0$

$\Rightarrow$ $\lambda = 0, \dfrac{30}{17}$ **1**

When $\lambda = 0$, x (– 2, – 1, 3)

When $\lambda = \dfrac{30}{17}$,

$x = 3 \times \dfrac{30}{17} - 2$

$= \dfrac{56}{17}$

$y = 2 \times \dfrac{30}{17} - 1$ **1**

$= \dfrac{43}{17}$

$z = 2 \times \dfrac{30}{17} + 3 = \dfrac{111}{17}$

$\therefore$ The points are (– 2, – 1, 3) and $\left(\dfrac{56}{17}, \dfrac{43}{17}, \dfrac{111}{17}\right)$ **1**

IInd Method :

A general point on the line

$$\frac{x+2}{3} = \frac{y+1}{2} = \frac{z-3}{2} = \lambda \text{ is}$$

$x = 3\lambda - 2$, $y = 2\lambda - 1$, $z = 2\lambda + 3$ **1**

Its distance from (1, 2, 3) = $3\sqrt{2}$

$\therefore$ $(3\sqrt{2})^2 = (3\lambda - 3)^2 + (2\lambda - 3)^2 + (2\lambda)^2$ **1**

$\Rightarrow$ $\lambda = 0$ or $\lambda = \dfrac{30}{17}$ **1**

$\therefore$ The points are $\left(\dfrac{56}{17}, \dfrac{43}{17}, \dfrac{111}{17}\right)$

or (–2, – 1, 3) **1**

22. Let E_1, E_2 and A be the events defined as follows :

E_1 = six occurs

E_2 = six does not occur

and A = the man reports that it is a six

We have

$P(E_1) = \dfrac{1}{6}$ and $P(E_2) = 1 - \dfrac{1}{6} = \dfrac{5}{6}$ **1**

Now,

$P\left(\dfrac{A}{E_1}\right)$ = Probability that the man speak truth $= \dfrac{3}{4}$

and $P\left(\dfrac{A}{E_2}\right)$ = Probability that the man does not speak truth

$= 1 - \dfrac{3}{4} = \dfrac{1}{4}$ **1**

We have to find $P\left(\dfrac{E_1}{A}\right)$ *i.e.,* the probability that there is six on the dice given that the

man has reported that there is six. By baye's theorem, we have

$$P\left(\frac{E_1}{A}\right) = \frac{P(E_1) \times P\left(\frac{A}{E_1}\right)}{P(E_1) \times P\left(\frac{A}{E_1}\right) + P(E_2) \times P\left(\frac{A}{E_2}\right)} \quad \textbf{1}$$

$$= \frac{\frac{1}{6} \times \frac{3}{4}}{\frac{1}{6} \times \frac{3}{4} + \frac{5}{6} \times \frac{1}{4}}$$

$$= \frac{3}{3+5} = \frac{3}{8} \quad \textbf{1}$$

Section–C

23.

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} \quad \textbf{1}$$

$$4A = 4\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 8 & 8 \\ 8 & 4 & 8 \\ 8 & 8 & 4 \end{bmatrix}$$

$$5I = 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \textbf{1}$$

$$= \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$\therefore\ A^2 - 4A - 5I$

$$= \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} - \begin{bmatrix} 4 & 8 & 8 \\ 8 & 4 & 8 \\ 8 & 8 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} \quad \textbf{1}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$= 0.$

Hence proved

Now, $|A| = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$

$= 1(1-4) - 2(2-4) + 2(4-2)$

$= -3 - 4 + 4 = -3 \neq 0$

$\therefore\ A^{-1}$ exists. **1**

From (1), we have

$$A^2 - 4A = 5I$$

$$\Rightarrow A^{-1}A^2 - 4A^{-1}A = 5A^{-1}I$$

$$\Rightarrow A - 4I = 5A^{-1}$$

$$\Rightarrow A^{-1} = \frac{1}{5}(A - 4I)$$

$$= \frac{1}{5}\left\{\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right\}$$

$$= \frac{1}{5}\left\{\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} - 4\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}\right\} \quad \textbf{1}$$

$$= \frac{1}{5}\begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{3}{5} & \frac{2}{5} & \frac{2}{5} \\ \frac{2}{5} & -\frac{3}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{2}{5} & -\frac{3}{5} \end{bmatrix} \quad \textbf{1}$$

OR

The given system of equations can be written as

$$AX = B,$$

where $A = \begin{bmatrix} 1 & 1 & -1 \\ 3 & 1 & -2 \\ 1 & -1 & -1 \end{bmatrix}$

$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$ **1**

Now $\quad |A| = \begin{vmatrix} 1 & 1 & -1 \\ 3 & 1 & -2 \\ 1 & -1 & -1 \end{vmatrix}$

$= 1(-1-2) - 1(-3+2) - 1(-3-1)$

$= -3 + 1 + 4$

$= 2$ **1**

$\neq 0.$

So, the given system of equations, has a unique solution given by

$$X = A^{-1} B.$$

Let C_{ij} be the co-factors of elements a_{ij} in A $= [a_{ij}]$, then

$C_{11} = -3$, $C_{12} = 1$, $C_{13} = -4$

$C_{21} = 2$, $C_{22} = 0$, $C_{23} = 2$

$C_{31} = -1$, $C_{32} = -1$, $C_{33} = -2$ **1**

$$\therefore \quad \text{adj } A = \begin{vmatrix} -3 & 2 & -1 \\ 1 & 0 & -1 \\ -4 & 2 & -2 \end{vmatrix}$$

and $\quad A^{-1} = \dfrac{1}{|A|} . \text{adj } A$

$$= \frac{1}{2}\begin{vmatrix} -3 & 2 & -1 \\ 1 & 0 & -1 \\ -4 & 2 & -2 \end{vmatrix}$$

Now $\quad X = A^{-1} B$

$$= \frac{1}{2}\begin{vmatrix} -3 & 2 & -1 \\ 1 & 0 & -1 \\ -4 & 2 & -2 \end{vmatrix}\begin{vmatrix} 1 \\ 3 \\ -1 \end{vmatrix} \quad \textbf{1}$$

$$\Rightarrow \begin{vmatrix} x \\ y \\ z \end{vmatrix} = \frac{1}{2}\begin{vmatrix} -3+6+1 \\ 1+0+1 \\ -4+6+2 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 4 \\ 2 \\ 4 \end{vmatrix}$$

$$\Rightarrow \begin{vmatrix} x \\ y \\ z \end{vmatrix} = \begin{vmatrix} 2 \\ 1 \\ 2 \end{vmatrix} \quad \textbf{1}$$

$\Rightarrow \quad x = 2, y = 1, z = 2.$

Hence $x = 2$, $y = 1$ and $z = 2$ is the required solution. **1**

24. Given equation of curve be

$$16x^2 + 9y^2 = 144 \quad \text{...(i)}$$

Since (x_1, y_1) lies on (i), then

$$16x_1^2 + 9y_1^2 = 144$$

$\Rightarrow \; 9y_1^2 = 144 - 16 \times 4, \quad$ [as $x_1 = 2$]

$\Rightarrow \quad 9y_1^2 = 80$

$\Rightarrow \quad y_1 = \pm\dfrac{4\sqrt{5}}{3}$ **1**

$\Rightarrow \quad y_1 = \dfrac{4\sqrt{5}}{3}, \quad$ [as $y_1 > 0$]

$\therefore \quad (x_1, y_1) = \left(2, \dfrac{4\sqrt{5}}{3}\right)$

Now, differentiating (i) w.r.t.x, we get

$$32x + 18y\frac{dy}{dx} = 0 \quad \textbf{1}$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{16x}{9y}$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = -\frac{16 \times 2}{9 \times \frac{4\sqrt{5}}{3}} = -\frac{8}{3\sqrt{5}}$$

$\therefore$ The equation of tangent at (x_1, y_1) is

$$y - y_1 = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} (x, x_1)$$

i.e., $\quad y - \dfrac{4\sqrt{5}}{3} = -\dfrac{8}{3\sqrt{5}}(x-2)$ **1**

$\Rightarrow \quad 3\sqrt{5}\, y - 20 = -8x + 16$

$\Rightarrow \; 8x + 3\sqrt{5}\, y - 36 = 0 \quad$ ...(ii)

Suppose tangent (ii) meets the x-axis at the point $(k, 0)$, then

$$8k + 0 - 36 = 0 \Rightarrow k = \frac{36}{8} = \frac{9}{2}$$

Thus, tangent intersect x-axis at the point $\left(\dfrac{9}{2}, 0\right)$ **1**

Again, equation of normal at (x_1, y_1) is

$$(y - y_1)\left(\frac{dy}{dx}\right)_{(x_1, y_1)} + (x - x_1) = 0$$

i.e., $\left(y - \dfrac{4\sqrt{5}}{3}\right) \times \left(-\dfrac{8}{3\sqrt{5}}\right) + (x - 2) = 0$

$\Rightarrow \; -8(3y - 4\sqrt{5}) + 9\sqrt{5}\,(x-2) = 0$

$\Rightarrow \; 9\sqrt{5}\, x - 24y + 14\sqrt{5} = 0 \quad$ ...(iii) **1**

Suppose normal (iii) meets the x-axis at the point $(m, 0)$, then

$$9\sqrt{5}\, m - 0 + 14\sqrt{5} = 0$$

$\Rightarrow \quad m = -\frac{14}{9}$

Thus, normal intersect the x-axits at the point $\left(\frac{-14}{9}, 0\right)$ **1**

OR

Let $\quad OC = h, CQ = r$

Now $\quad OA = R$, (Given)

Height of the cone $= OC + OA = h + R$

$\therefore \quad V =$ volume of the cone

$= \frac{1}{3}\pi r^2 (h + R) \quad ...(i)$ **1**

Also, from ΔOCD,

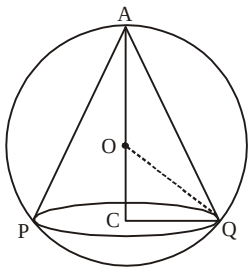
$OC^2 + CQ^2 = OQ^2$

$\Rightarrow \quad h^2 + r^2 = R^2$

$\Rightarrow \quad r_2^2 = R^2 - h^2 \quad ...(ii)$ **1**

From (i) and (ii), we get

$V = \frac{1}{3}\pi (R^2 - h^2)(h + R) \quad ...(iii)$



Differentiating w.r.t. h, we get

$\frac{dV}{dh} = \frac{1}{3}\pi[-2h(h + R) + (R^2 - h^2).1]$ **1**

$= \frac{1}{3}\pi[R^2 - 2hR - 3h^2]$

and $\frac{d^2V}{dh^2} = \frac{1}{3}\pi[-2R - 6h]$

For maximum or minimum, **1**

$\frac{dV}{dh} = 0$

$\Rightarrow R^2 - 2hR - 3h^2 = 0$

$\Rightarrow (R + h)(R - 3h) = 0$

$\Rightarrow \quad h = \frac{1}{3}R, \ (h \neq -R)$

For this value of h, we get

$\frac{d^2V}{dh^2} = -\frac{4}{3}\pi R < 0$

$\therefore$ V is maximum and from (iii) it is given by

$V = \frac{1}{3}\pi\left(R^2 - \frac{R^2}{9}\right)\left(\frac{1}{3}R + R\right)$

$= \frac{1}{3}\pi\frac{8R^2}{9}.\frac{4R}{3} = \frac{8}{27}\left(\frac{4}{3}\pi R^3\right)$

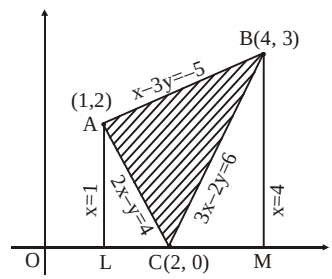
$= \frac{8}{27} \times$ Volume of the sphere **1**

25. The given lines are

$2x + y = 4 \quad ...(1)$

$3x - 2y = 6 \quad ...(2)$

$x - 3y = -5 \quad ...(3)$



Multiply (3) by 2 and subtract from (1)

$7y = 14$

$y = 2$

From (3), $x - 3 \times 2 = -5$

$\Rightarrow \quad x = 1$

Lines (1) and (3) intersect at (1, 2)

Solving (3) and (2), we get $x = 4, y = 3$

Lines (2) and (3) intersect at (4, 3)

Solving (1) and (2), we get $x = 2, y = 0$ **1**

Lines (1) and (2) intersect at (2, 0)

Area of ΔABC = Area of trapezium ALMB – Area of ΔALC – Area of ΔBCM

$= \int_1^4 y_1\,dx - \int_1^4 y_2\,dx - \int_2^4 y_3\,dx$

where y_1 is for $x - 3y = -5$ or $y = \frac{x+5}{3}$ **1**

y_2 is for $2x + y = 4$ or $y = 4 - 2x$

y_3 is for $3x - 2y = 6$ or $y = \frac{3x-6}{2}$

Area of ΔABC

$= \int_1^4 \frac{x+5}{3}dx - \int_1^2 (4 - 2x)\,dx - \int_2^4 \frac{3x-6}{2}dx$ **1**

$= \frac{1}{3}\left[\frac{x^2}{2} + 5x\right]_1^4 - [4x - x^2]_1^2 - \frac{1}{2}\left[\frac{3x^2}{2} - 6x\right]_2^4$

$$= \frac{1}{3}\left[\left(\frac{16}{2}+20\right)-\left(\frac{1}{2}+5\right)\right]-[(8-4)-(4-1)]$$

$$-\frac{1}{2}\left[\left(\frac{3\times 16}{2}-24\right)-\left(\frac{12}{2}-12\right)\right]$$

$$= \frac{1}{3}\left[28-\frac{11}{2}\right]-[4-3]-\frac{1}{2}[0+6]$$

$$= \left[\frac{1}{3}\times\frac{45}{2}-1-3\right]$$

$$= \frac{15}{2}-4$$

$$= \frac{7}{2}\text{ sq. units}$$ **1**

26. Let $I = \int_0^{\pi/2}\left(\sqrt{\tan x}+\sqrt{\cot x}\right)dx$

Put, $\sqrt{\tan x} = t \Rightarrow \tan x = t^2$

$\Rightarrow \sec^2 x\, dx = 2t\, dt \Rightarrow dx = \dfrac{2t}{1+t^4}\,dt$ **1**

Also, $x = 0 \Rightarrow t = 0$ and $x = \dfrac{\pi}{2} \Rightarrow t \to \infty$

$\therefore\ I = \int_0^{\infty}\left(t+\dfrac{1}{t}\right)\dfrac{2t}{1+t^4}\,dt$ **1**

$$= 2\int_0^{\infty}\frac{t^2+1}{1+t^4}\,dt$$

$$= 2\int_0^{\infty}\frac{1+\frac{1}{t^2}}{t^2+\frac{1}{t^2}}\,dt \qquad ...(1)$$

Put, $t-\dfrac{1}{t} = y \Rightarrow \left(1+\dfrac{1}{t^2}\right)dt = dy$

Also, $t = 0 \Rightarrow y \to -\infty$ and $t \to \infty \Rightarrow y \to \infty$

and $t^2+\dfrac{1}{t^2} = \left(t-\dfrac{1}{t}\right)^2 + 2 = y^2+2$ **1**

$\therefore$ From (1), we get

$$I = 2\int_{-\infty}^{\infty}\frac{dy}{y^2+2} = 2.2\int_0^{\infty}\frac{dy}{y^2+\left(\sqrt{2}\right)^2}$$ **1**

$$= 4.\frac{1}{\sqrt{2}}\left[\tan^{-1}\frac{y}{\sqrt{2}}\right]_0^{\infty}$$ **1**

$$= 2\sqrt{2}\left[\frac{\pi}{2}-0\right] = \pi\sqrt{2}$$ **1**

27. The given planes are

$2x + y + 3z - 2 = 0$ ...(1)

and $x - 2y + 5 = 0$ ...(2)

Direction ratios of normal of plane (1) are 2, 1, 3. **1**

Directions ratios of normal of plane (2) are 1, 2, 0. **1**

Planes (1) and (2) are perpendicular to each other, then **2**

$a_1a_2 + b_1b_2 + c_1c_2 = 0$ *i.e.,* $2.1 + 1.(-2) + 3.0$
$= 2 - 2 + 0 = 0$ **1**

Hence, the planes (1) and (2) are perpendicular to each other. **1**

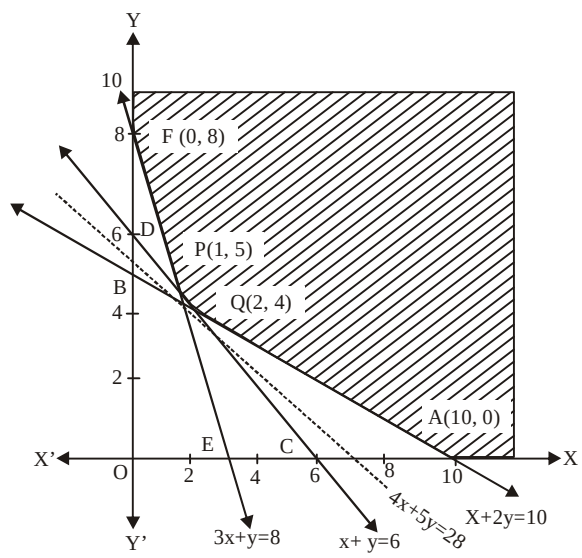
28. Let there be x kg food X and y food Y.

Food X cots Rs. 16 and food Y costs Rs. 20 per kg.

$\therefore$ $Z = 16x + 20y$ is the objective function, where is to be minimize.

Now constraints are

$x + 2y \geq 10$, $2x + 2y \geq 12$ or $x + y \geq 6$ and $3x + y \geq 8x$, $y \geq 0$.



1

(i) The line $x + 2y = 10$ passes through A (10, 0) and B (0, 5). Putting $x = 0$, $y = 0$ in $x + 2y \geq 10$.

$0 \geq 10$ which is not true.

$\Rightarrow$ $x + 2y \geq 10$ region lies on and above AB.

(ii) The line $x + y = 6$ passes through C(6, 0) and D(0, 6).

Putting $x = 0$, $y = 0$ *i.e.,* $x + y \geq 6$.

$0 \geq 6$ which is not true.

$\Rightarrow$ $x + y \geq 6$ lies on and above CD. **1**

(iii) The line $3x + y = 8$ passes through E $\left(\dfrac{8}{3}, 0\right)$, F(0, 8).

Putting $x = 0$, $y = 0$ in $3x + y \geq 8$ or $0 \geq 8$ which is not true.

$\Rightarrow 3x + y \geq 8$ lies on and above EF.

(iv) $x \geq 0$ lies on and to the right of y-axis.

(v) $y \geq 0$ lies on and above x-axis. **1**

The shaded area YFPQAY represents the feasible region. Now, P is the point of intersection of the lines CD and EF.

$$x + y = 6 \quad ...(1)$$
$$3x + y = 8 \quad ...(2)$$

Subtracting (1) from (2)

$$2x = 2, x = 1, y = 5$$

$\therefore$ The point P is (1, 5).

Q is the point of intersection of the line

$$x + 2y = 10 \quad ...(1)$$
$$x + y = 6 \quad ...(2)$$

Subtracting (2) from (1),

$$y = 4, x = 2$$

The objective function is

$$Z = 16x + 20y$$

At F(0, 8), $Z = 16x + 20y = 0 + 160 = 160$

At P(1, 5), $Z = 16 + 100 = 116$

At Q(2, 4), $Z = 32 + 80 = 112$

At A(10, 0), $Z = 160 + 0 = 160$ **1**

This shows that minimum value of Z is Rs. 112.

Feasible region is unbounded.

Consider the inequality $16x + 20y < 112$

or $4x + 5y = 28$

The line $4x + 5y = 28$ passes through (7, 0) and $\left(0, \frac{28}{5}\right)$.

Also, putting $x = 0$, $y = 0$ is

$$4x + 5y < 28$$

we get, $0 < 28$ which is true. **1**

$\therefore$ $8x + 10y = 33$ lie below the line $4x + 5y = 28$

Hence, there is no common point between feasible region and $4x + 5y < 33$.

$\therefore$ Mimimum value of $Z = 112$, $x = 2$, $y = 4$ *i.e.,* least value of mixture of Rs. 112 when 2 kg of food X and 4 kg of food Y are mixed. **1**

29. Total number of balls = 25

Number of balls marked 'X' = 10

Let X denote the event of getting a ball marked X.

Y = event of getting a ball Y

$$\therefore \quad P(X) = \frac{10}{25} = \frac{2}{5} = p$$

$$\therefore \quad P(Y) = 1 - \frac{2}{5} = \frac{3}{5} = q$$

Now 6 balls are drawn. **1**

(i) P(all the balls are marked X) = $\left(\frac{2}{5}\right)^6$

(ii) Event that not more than 2 will bear 'Y' mark)

$\Rightarrow$ (6X, 0Y), (5X, 1Y), (4X, 2Y)

$\therefore$ P(not more than 2 will bear the mark 'Y')

$$= \left(\frac{2}{5}\right)^6 + {}^6C_5 \left(\frac{3}{5}\right)\left(\frac{2}{5}\right)^5 + {}^6C_4 \left(\frac{3}{5}\right)^2\left(\frac{2}{5}\right)^4$$

$$= \left(\frac{2}{5}\right)^4 \left(\frac{4}{25} + \frac{6 \times 3 \times 2}{25} + \frac{15 \times 9}{25}\right) \quad \textbf{1}$$

$$= \left(\frac{2}{5}\right)^4 \left(\frac{4 + 36 + 135}{25}\right)$$

$$= \left(\frac{2}{5}\right)^4 \left(\frac{175}{25}\right) = 7\left(\frac{2}{5}\right)^4$$

(iii) Event that at least 1 ball will bear 'Y'

= [(5X, 1Y), (4X, 2Y), (3X, 3Y), (2X, 4Y), (X, 5Y), (0X, 6Y)} **1**

P(at least 1 ball will bear mark Y)

= 1 – P(no ball will bear mark Y)

= 1 – P(all balls bear mark Y)

$$= 1 - \left(\frac{2}{5}\right)^6 \quad \textbf{1}$$

(iv) P(3 balls each are marked X and Y)

$$P(3) = {}^6C_3 \left(\frac{3}{5}\right)^3 \left(\frac{2}{5}\right)^3$$

$$= \frac{6 \times 5 \times 4}{6} \times \frac{27}{125} \times \frac{8}{125}$$

$$= \frac{864}{3125} \quad \textbf{1}$$

●●

Sample Question Paper—9

Section–A

1. R : {(x, y) : x is wife of y}

R is not reflexive as x cannot be wife of x.

R is not symmetric as x is wife of y but y is not wife of x.

R is not transitive as if x is a wife of y, then y cannot be the wife of anybody else. **1**

2. $$\cos^{-1}\left(\frac{1}{2}\right) = 2\sin^{-1}\left(\frac{1}{2}\right)$$

Now $\cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$, $\sin^{-1}\frac{1}{2} = \frac{\pi}{6}$

$$\therefore \cos^{-1}\left(\frac{1}{2}\right) + 2\sin^{-1}\left(\frac{1}{2}\right)$$

$$= \frac{\pi}{3} + \frac{2\pi}{6} = \frac{2\pi}{3}$$ **1**

3. (i) $$\begin{bmatrix} a & b \\ -b & a \end{bmatrix} + \begin{bmatrix} a & b \\ b & a \end{bmatrix}$$

$$= \begin{bmatrix} a+a & b+b \\ -b+b & a+a \end{bmatrix} = \begin{bmatrix} 2a & 2b \\ 0 & 2a \end{bmatrix}$$ **1**

4. (i) For a symmetric matrix A' = A.

$$A' = \begin{bmatrix} 1 & -1 & 5 \\ -1 & 2 & 1 \\ 5 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 5 \\ -1 & 2 & 1 \\ 5 & 1 & 3 \end{bmatrix} = A$$

$\therefore$ A' = A $\Rightarrow$ A is symmetric matrix. **1**

5. Given,

$$\begin{bmatrix} x & 2 \\ 18 & x \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 18 & 6 \end{bmatrix}$$

From above on comparing, **1**

$$x = 6$$

6. The rate of change of the circle w.r.t. time t is given to be 0.7 cm/sec *i.e.,*

$$\frac{dr}{dt} = 0.7 \text{ cm/sec.}$$

Now, circumference of the circle is C = $2\pi r$.

$\therefore$ The rate of change of circumference w.r.t. t is given by

$$\frac{dC}{dt} = \left(\frac{d}{dr}(2\pi r).\frac{dr}{dt}\right)$$ **½**

$$= 2\pi \cdot \frac{dr}{dt} = 2\pi \times 0.7$$

$$= 1.4\pi \text{ cm/sec.}$$ **½**

7. $$\int (ax^2 + bx + c)\, dx$$

$$= \int ax^2\, dx + \int bx\, dx + \int c\, dx$$

$$= \frac{ax^3}{3} + \frac{bx^2}{2} + cx + C$$ **1**

8. $\vec{P}$ (1, 5, 4), $\vec{Q}$ (4, 1, – 2)

$$\vec{PQ} = \vec{OQ} - \vec{OP}$$

$= (4\hat{i} + \hat{j} - 2\hat{k}) - (\hat{i} + 5\hat{j} - 4\hat{k})$ **½**

$= (3\hat{i} + 4\hat{j} - 6\hat{k})$

$\therefore$ Direction of $\vec{PQ}$ are (3, – 4, – 6) **½**

9. $$|\vec{a}| = \sqrt{3}$$

$$|\vec{b}| = 2,$$

$$\vec{a} \cdot \vec{b} = 3$$

$$\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos\theta$$

$$\Rightarrow \cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|\,|\vec{b}|}$$ **½**

$$\Rightarrow \cos\theta = \frac{3}{\sqrt{3} \times 2}$$

$$\Rightarrow \cos\theta = \frac{\sqrt{3}}{2}$$

$$\Rightarrow \cos\theta = \cos 60^\circ$$

$$\Rightarrow \theta = 60^\circ$$ **½**

10. The equation of a plane through the line of intersection of planes

$x + 2y + 3z - 4 = 0$ and $2x + y - z + 5 = 0$ is

$(x + 2y + 3z - 4) + \lambda(2x + y - z + 5) = 0$

$x(1 + 2\lambda) + y(2 + \lambda) + z(3 - \lambda) - 4 + 5\lambda = 0$ ...(1)

This is perpendicular to the plane

$5x + 3y + 6z + 8 = 0$

$\therefore \quad 5(1 + 2\lambda) + 3(2 + \lambda) + 6(3 - \lambda) = 0$

$7\lambda + 29 = 0$

$\lambda = -\dfrac{29}{7}$ **½**

Putting $\lambda = -\dfrac{29}{7}$ in (1), we obtain the equation of the required plane as

$-51x - 15y + 50z - 173 = 0$

$51x + 15y - 50z + 173z = 0.$ **½**

Section–B

11. The set A

$\{x \in Z : 0 \le x \le 12\} = \{0, 1, 2, ..., 12\}$

(i) $R = \{(a, b) : |a - b| \text{ is a multiple of } 4\}$

$|a - b| = 4k$ or $b = a + 4k$

$\therefore$ R = {(1, 5), (1, 9), (2, 6), (2, 10), (3, 7), (3, 11), (4, 8), (4, 12), (5, 9), (6, 10), (7, 11), (8, 12), (0, 0), **1**

(1, 1), (2, 2), ..., (12, 12)

(a) $a - a = 0 = 4k$, where $k = 0 \Rightarrow (a, a) \in R$

$\therefore$ R is reflexive.

(b) If $|a - b| = 4k$, then $|b - a| = 4k$ *i.e.*, (a, b) and (b, a) both belong to R. R is symmetric. **1**

(c) $a - c = a - b + b - c$

When $a - b$ and $b - c$ both are multiples of 4 then $a - c$ is also multiple of 4. This shows if $(a, b), (b, c) \in R$, then $a - c$ also $\in R$.

$\therefore$ R is an equivalence relation.

The sets related 1 are {(1, 5), (1, 9)}.

(ii) $R = \{(a, b) : a = b\}$

$\therefore$ {(0, 0), (1, 1), (2, 2) ..., (12, 12)}

(a) $a = a \Rightarrow (a, a) \in R$ **1**

$\therefore$ R is reflexive.

(b) Again if $(a, b) \in R$, then (b, a) also $\in R$, Since $a = b$ and $(a, b) \in R \Rightarrow R$ is symmetric.

(c) If $(a, b) \in R$, then $(b, c) \in R \Rightarrow a = b = c$

$\therefore \quad a = c \Rightarrow (a, c) \in R$. Hence R is transitive. **1**

Set related to 1 is {1}.

12. We have

$$1 + \sin x = \cos^2\frac{x}{2} + \sin^2\frac{x}{2} + 2\sin\frac{x}{2}\cos\frac{x}{2}$$

$$= \left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)^2$$

Similarly,

$$1 - \sin x = \left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)^2 \quad \textbf{1}$$

Now $\quad \cot^{-1}\left(\dfrac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}}\right)$ **1**

$$= \cot^{-1}\left(\frac{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right) + \left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)}{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right) - \left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)}\right)$$

$$= \cot^{-1}\left(\frac{2\cos\frac{x}{2}}{2\sin\frac{x}{2}}\right) \quad \textbf{1}$$

$$= \cot^{-1}\left(\cot\frac{x}{2}\right)$$

$$= \frac{x}{2} \quad \textbf{1}$$

13. $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ bc & ca & ab \end{vmatrix}$, $C_1 - C_3$ and $C_2 - C_3$

$$\Delta = \begin{vmatrix} 0 & 0 & 1 \\ a-c & b-c & c \\ b(c-a) & a(c-b) & ab \end{vmatrix} \quad \textbf{1}$$

$$= (a - c)(b - c)\begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & c \\ -b & -a & ab \end{vmatrix} \quad \textbf{1}$$

$\Delta = (a - c)(b - c)[(-a) - (-b)]$

$= (a - c)(b - c)(b - a)$ **1**

$\Delta = (a - b)(b - c)(c - a).$ **1**

OR

Consider, $\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix}$

$$= \begin{vmatrix} 2(a+b+c) & a & b \\ 2(a+b+c) & b+c+2a & b \\ 2(a+b+c) & a & c+a+2b \end{vmatrix} \quad \textbf{1}$$

[By performing $C_1 \to C_1 + (C_2 + C_3)$]

$$= 2(a + b + c)\begin{vmatrix} 1 & a & b \\ 1 & b+c+2a & b \\ 1 & a & c+a+2b \end{vmatrix}$$

[By taking $2(a + b + c)$ common from C_1] **1**

$$= 2(a + b + c)\begin{vmatrix} 1 & a & b \\ 0 & b+c+a & 0 \\ 0 & 0 & a+b+c \end{vmatrix}$$

[By performing $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$]

$$= 2(a+b+c)\begin{vmatrix} 1 & a & b \\ 0 & b+c+a & 0 \\ 0 & 0 & a+b+c \end{vmatrix} \quad \mathbf{1}$$

[By performing $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$]
Expanding along C_1, we get
$= 2(a + b + c)\,[(a + b + c)^2 - 0]$
$= 2(a + b + c)^3$.

14. Let $y = (\sin x)^x + \sin^{-1}\sqrt{x}$

$= u + v$

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

Now $u = (\sin x)^x$ **1**

Taking log of both sides

$\log u = \log (\sin x)^x$

$= x \log (\sin x)$,

$[\because \log m^n = n \log m]$

Differentiating both sides w.r.t. x

$$\frac{1}{u}\frac{du}{dx} = 1.\log \sin x + x.\frac{1}{\sin x}\frac{d}{dx}(\sin x) \quad \mathbf{1}$$

$[(uv) = u'v + uv', u = x', v = \log (\sin x)]$

$$= \log \sin x + \frac{x\cos x}{\sin x}$$

$$\therefore \quad \frac{du}{dx} = u\,[\log \sin x + x \cot x]$$

$= (\sin x)^x\,[\log \sin x + x \cot x]$

$v = \sin^{-1}\sqrt{x}$

Put $v = \sin^{-1} t$, $t = \sqrt{x}$

$$\therefore \quad \frac{dv}{dt} = \frac{1}{\sqrt{1-t^2}}, \frac{dt}{dx} = \frac{1}{2}x^{1/2-1} = \frac{1}{2\sqrt{x}} \quad \mathbf{1}$$

$$\therefore \quad \frac{dv}{dx} = \frac{dv}{dt} \times \frac{dt}{dx} = \frac{1}{\sqrt{1-t^2}}.\frac{1}{2\sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}\sqrt{1-x}}$$

$$\because \quad \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

$$\Rightarrow \frac{dy}{dx} = (\sin x)^x\,[\log \sin x + x \cot x] + \frac{1}{2\sqrt{x-x^2}} \quad \mathbf{1}$$

15. Let $P(n)$ be the given statement in the problem.

$$\therefore \quad P(n) : \frac{d}{dx}(x^n) = nx^{n-1} \quad ...(1)$$

To verify the statement for $n = 1$.
Put $n = 1$ in (1), we get

$$P(1) : \frac{d}{dx}(x^1) = (1)x^{1-1} = (1)\,(x)^0$$

$$= (1)\,(1) = 1 \quad \mathbf{1}$$

which is true as $\frac{d}{dx}(x) = 1$.

We suppose $P(x)$ is true for $n = m$.

$$\therefore \quad P(m) : \frac{d}{dx}(x^m) = mx^{m-1} \quad ...(2) \ \mathbf{1}$$

To establish the truth of $P(m + 1)$, we prove

$$P(m+1) : \frac{d}{dx}(x^{m+1}) = (m+1)\,x^m$$

Since $x^{m+1} = x^1.x^m$

$$\therefore \quad \frac{d}{dx}(x^{m+1}) = \frac{d}{dx}(x.x^m)$$

$$= x.\frac{d}{dx}(x^m) + x^m.\frac{d}{dx}(x) \quad \mathbf{1}$$

$= x.mx^{m-1} + x^m.1$, (Using (2)]

$= mx^m + x^m = (m + 1)x^m$

$= (m + 1)_x^{(m+1)-1}$

$\therefore$ $P(m + 1)$ is true, if $P(m)$ is true. But $P(1)$ is true.

$\therefore$ By principle of induction, $P(n)$ is true for all $n \in N$. **1**

16. The given curves are

$x = y^2$ ...(1)

and $xy = k$ ...(2)

These meet where

$y^2.y = k \Rightarrow y^3 = k \Rightarrow y = k^{1/3}$

and $x = y^2 = k^{2/3}$ **1**

i.e., at the point $P(k^{2/3}, k^{1/3})$

Differentiating (1) w.r.t. x, we get

$$1 = 2y\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{2y}$$

$\therefore$ m_1 = Slope of tangent to the curve (1) at

$$P = \left(\frac{dy}{dx}\right)_{\text{at P}} = \frac{1}{2k^{1/3}} \quad \mathbf{1}$$

Similarly, differentiating (2) w.r.t. x, we get

$$x\frac{dy}{dx} + 1.y = 0 \Rightarrow \frac{dy}{dx} = -\frac{y}{x}$$

$\therefore$ m_2 = slope of tangent to the curve (2) at P

$$= \left(\frac{dy}{dx}\right)_{\text{at P}} = -\frac{k^{1/3}}{k^{2/3}} \quad \mathbf{1}$$

For the curves to cut at right angle at P, we must have

$$m_1 m_2 = -1$$

$$\Rightarrow \quad \frac{1}{2k^{1/3}} \times \left(-\frac{k^{1/3}}{k^{2/3}}\right) = -1$$

$$\Rightarrow \quad 2k^{2/3} = 1$$

$$\Rightarrow \quad 2^3 k^2 = (1)^3$$

$$\Rightarrow \quad 8k^2 = 1. \qquad \textbf{1}$$

OR

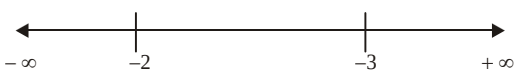
Given,

$$f(x) = 2x^3 - 3x^2 - 36x + 7$$

$$\Rightarrow \quad f'(x) = 6x^2 - 6x - 36 = 6(x^2 - x - 6) \qquad \textbf{1}$$

$$= 6(x - 3)(x + 2)$$

$\therefore$ $f'(x) = 0$ gives $x = 3, -2$. The points $x = -2$ and $x = 3$ divides the real line into three disjoint intervals, namely, $(-\infty, -2)$, $(-2, 3)$ and $(3, \infty)$.



In the intervals $(-\infty, -2)$ and $(3, \infty)$, $f'(x)$ is positive while in the interval $(-2, 3)$, $f'(x)$ is negative. Consequently, the function f' is strictly increasing in the intervals $(-\infty, 2)$ and $(3, \infty)$ while the function is strictly decreasing in the intervals $(-2, 3)$. **2**

17. Let $\quad I = \int \frac{2\sin x + 3\cos x}{3\sin x + 4\cos x}\, dx$

Write $2 \sin x + 3 \cos x$

$$= \lambda \frac{d}{dx}(3 \sin x + 4 \cos x) + \mu (3 \sin x + 4 \cos x) \qquad \textbf{1}$$

$$= \lambda (3 \cos x - 4 \sin x) + \mu (3 \sin x + 4 \cos x)$$

$$\Rightarrow \quad 2 = -4\lambda + 3\mu$$

$$3 = 3\lambda + 4\mu \qquad \textbf{1}$$

Solving these, we get

$$\lambda = \frac{1}{25}, \ \mu = \frac{18}{25} \qquad ...(1)$$

Hence,

$$I = \int \frac{\lambda(3\cos x - 4\sin x) + \mu(3\sin x + 4\cos x)}{3\sin x + 4\cos x}\, dx \qquad \textbf{1}$$

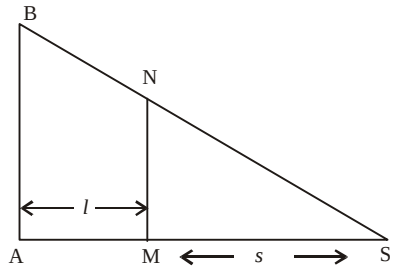
$$= \lambda \int \frac{3\cos x - 4\sin x}{3\sin x + 4\cos x}\, dx + \mu \int dx + c$$

$$= \frac{1}{25} \log |3 \sin x + 4 \cos x| + \frac{18}{25} x + c,$$

put $3 \sin x + 4 \cos x = 0$ **1**

OR

In the figure, let AB be the lamp-post, the lamp being at the position B and let MN be the man at a particular time t and let AM = l metres. Then MS is the shadow of the man. Let MS = s metres. **1**



Note that Δ MSN $\cong$ Δ ASB

or $\quad \frac{MS}{AS} = \frac{MN}{AB}$ **1**

or AS = $3s$, [as MN = 2 and AB = 6 (given)]

Thus AM = $3s - s = 2s$. But AM = l

So, $l = 2s$

Therefore $\frac{dl}{dt} = 2\frac{ds}{dt}$ **1**

Since $\frac{dl}{dt} = 5$ km/h. Hence, the length of the shadow increases at the rate $\frac{5}{2}$ km/h. **1**

18. Let $\quad I = \int \frac{e^x}{\sqrt{5 - 4e^x - e^{2x}}}\, dx$

Put $\quad e^x = t$, so $e^x\, dx = dt$

$$I = \int \frac{dt}{\sqrt{5 - 4t - t^2}} \qquad \textbf{1}$$

$$= \int \frac{dt}{\sqrt{(3)^2 - (t+2)^2}} \qquad \textbf{1}$$

$$= \sin^{-1}\left(\frac{t+2}{3}\right) + c \qquad \textbf{1}$$

$$= \sin^{-1}\left(\frac{e^x + 2}{3}\right) + c. \qquad \textbf{1}$$

19. Let $\quad I = \int \frac{1}{5\cos x - 12\sin x}\, dx$

$$= \int \frac{dx}{5\left(\cos^2\frac{x}{2} - \sin^2\frac{x}{2}\right) - 12 \times 2\sin\frac{x}{2}\cos\frac{x}{2}}$$

$$= \int \frac{\sec^2\frac{x}{2}}{5 - 5\tan^2\frac{x}{2} - 24\tan\frac{x}{2}}\, dx$$

Dividing numerator and denominator by $\cos^2 \frac{x}{2}$

Let $\tan \frac{x}{2} = t$, so that $\frac{1}{2} \sec^2 \frac{x}{2}\, dx = dt$

$$\therefore \quad I = \int \frac{2}{5 - 5t^2 - 25t}\, dt$$

$$= \frac{1}{5} \int \frac{2dt}{1 - \left(t^2 + \frac{24}{5} t\right)} \qquad \textbf{1}$$

$$= \frac{2}{5} \int \frac{1}{\left(\frac{13}{5}\right)^2 - \left(t + \frac{12}{5}\right)^2}\, dt$$

$$= \frac{2}{5} \times \frac{1}{2 \times \frac{13}{5}} \log \left| \frac{\frac{13}{5} + \left(t + \frac{12}{5}\right)}{\frac{13}{5} - \left(t + \frac{12}{5}\right)} \right| + c \qquad \textbf{1}$$

$$= \frac{1}{13} \log \left| \frac{5 + \tan \frac{x}{2}}{\frac{1}{5} - \tan \frac{x}{2}} \right| + c$$

$$= \frac{1}{13} \log \left| \frac{25 + \tan \frac{x}{2}}{1 - 5 \tan \frac{x}{2}} \right| + c \qquad \textbf{1}$$

20. Given, $(\vec{a} + \vec{b}).(\vec{a} - \vec{b}) = 8$

$\vec{a}.(\vec{a} - \vec{b}) + \vec{b}.(\vec{a} - \vec{b}) = 8$

$\vec{a}.\vec{a} - \vec{a}.\vec{b} + \vec{b}.\vec{a} - \vec{b}.\vec{b} = 8$ **1**

$\Rightarrow |\vec{a}|^2 - |\vec{b}|^2 = 8,$

$[\because \vec{a}.\vec{b} = \vec{b}.\vec{a}]$

$\Rightarrow 64 |\vec{b}|^2 - |\vec{b}|^2 = 8 \; [\because |\vec{a}| = 8|\vec{b}|]$

or $63 |\vec{a}]^2 = 8 \therefore |\vec{b}|^2 = \frac{8}{63}$

But $|\vec{a}| = 8|\vec{b}|$

$$= 8 \times \sqrt{\frac{8}{63}} = \frac{8\sqrt{8}}{\sqrt{63}}$$

Thus, $|\vec{a}| = \frac{8\sqrt{8}}{\sqrt{63}}, |\vec{b}| = \sqrt{\frac{8}{63}}$ **1**

21. Let the direction ratios of the required line be *a, b, c*. Since, it is perpendicular to the two given lines.

Therefore, $a + 2b + 3c = 0$ ...(i) **1**

$-3a + 2b + 5c = 0$ ...(ii)

Solving (i) and (ii) by cross-multiplication, we get

$$\frac{a}{4} = \frac{b}{-14} = \frac{c}{8}$$

$$\frac{a}{2} = \frac{b}{-7} = \frac{c}{4} = k, \quad \text{(say)} \; \textbf{1}$$

Thus, the required line passes through $(-1, 3, -2)$ and has direction ratios proportional to $2, -7, 4$. So, its equation is **1**

$$\frac{x+1}{2} = \frac{y-3}{-7} = \frac{z+2}{4} \qquad \textbf{1}$$

OR

Equation of the plane through the point (3, 4, 2) is

$a(x - 3) + b(y - 4) + c(z - 2) = 0$ ...(i) **1**

Plane (i) passes through $(2, -2, -1)$ and $(7, 0, 6)$.

Therefore $a(-1) + b(-4) + c(4) = 0$ ...(ii)

and $a(4) + b(-4) + c(4) = 0$

or $a - b + c = 0$ ...(iii) **1**

Eliminating *a, b, c* from (i), (ii) and (iii), we get

$$\begin{vmatrix} x-3 & y-4 & z-2 \\ -1 & -6 & -3 \\ 1 & -1 & 1 \end{vmatrix} = 0$$

$\Rightarrow (x - 3)(-6 - 3) - (y - 4)(-1 + 3)$
$+ (z - 2)(1 + 6) = 0$ **1**

$\Rightarrow -9x + 27 - 2y + 8 + 7z - 14 = 0$
$9x + 2y - 7z - 21 = 0$

is the required equation. **1**

22. Let

E_1 = the event that bag I is chosen

E_2 = the event that bag II is chosen

and A = the event that a white ball is drawn

$$\therefore \quad P(E_1) = P(E_2) = \frac{1}{2} \qquad \textbf{1}$$

$$P\left(\frac{A}{E_1}\right) = \frac{3}{7}, \; P\left(\frac{A}{E_1}\right) = \frac{5}{11} \qquad \textbf{1}$$

By Baye's theorem, the required probability is

$$P\left(\frac{E_1}{A}\right) = \frac{P(E_1) \times P\left(\frac{A}{E_1}\right)}{P(E_1) \times P\left(\frac{A}{E_1}\right) + P(E_2) \times P\left(\frac{A}{E_2}\right)}$$

1

$$= \frac{\frac{1}{2}\times\frac{3}{7}}{\frac{1}{2}\times\frac{3}{7}+\frac{1}{2}\times\frac{5}{11}} = \frac{\frac{3}{7}}{\frac{3}{7}+\frac{5}{11}} = \frac{33}{68} \qquad \textbf{1}$$

Section–C

23. The given system of equations can be written as

$$AX = B$$

where $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ 2 & 1 & -3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and

$$B = \begin{bmatrix} 4 \\ -1 \\ -9 \end{bmatrix} \qquad \textbf{1}$$

Now $|A| = 1(3-1) - 1(-6-2) + 1(2+2)$ **1**

$= 2 + 8 + 4 = 14 \neq 0$

$\therefore$ A^{-1} exists and it is given by

$$A^{-1} = \frac{1}{|A|}\text{adj } A = \frac{1}{14}\begin{bmatrix} 2 & 4 & 2 \\ 8 & -5 & 1 \\ 4 & 1 & -3 \end{bmatrix} \qquad \textbf{1}$$

By equation (1), we get

$$X = A^{-1}B$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{14}\begin{bmatrix} 2 & 4 & 2 \\ 8 & -5 & 1 \\ 4 & 1 & -3 \end{bmatrix}\begin{bmatrix} 4 \\ -1 \\ -9 \end{bmatrix} \qquad \textbf{1}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{14}\begin{bmatrix} -14 \\ 28 \\ 42 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} \qquad \textbf{1}$$

Thus $x = -1, y = 2, z = 3.$

OR

Given $A = \begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix}$, then

$$A^2 = A.A = \begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix}\times\begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix} = \begin{bmatrix} 16 & 8 \\ 56 & 32 \end{bmatrix}$$

Also $I = I^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ **1**

$\therefore$ $A^2 + xI = yA$ ...(1)

$$\Rightarrow \begin{bmatrix} 16 & 8 \\ 56 & 32 \end{bmatrix} + x\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = y\begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 16+x & 8+0 \\ 56 & 32+x \end{bmatrix} = \begin{bmatrix} 3y & y \\ 7y & 5y \end{bmatrix} \qquad 1$$

$\Rightarrow$ $16 + x = 3y,\ 8 = y$

$56 = 7y,\ 32 + y = 5y$

$\Rightarrow$ $y = 8,\ x = 3y - 16 = 3 \times 8 - 16 = 8$

$\therefore$ $x = 8,\ y = 8.$

Putting these values of x and y in (1), we get

$A^2 + 8I = 8A$ ...(2) **1**

Now $|A| = \begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix} = 15 - 7 = 8 \neq 0$

$\therefore$ A^{-1} exists.

Pre-multiplying both sides of (2) by A^{-1}, we get

$A^{-1}A^2 + 8A^{-1}I = 8A^{-1}A$

$\Rightarrow$ $A + 8A^{-1} = 8I$

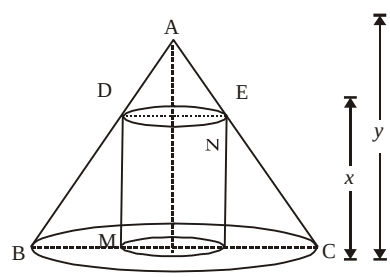
$\Rightarrow$ $8A^{-1} = 8I - A$ **1**

$$\Rightarrow 8A^{-1} = 8\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 8-3 & 0-1 \\ 0-7 & 8-5 \end{bmatrix} \qquad \textbf{1}$$

$$\therefore A^{-1} = \frac{1}{8}\begin{bmatrix} 5 & -1 \\ -7 & 3 \end{bmatrix} \qquad \textbf{1}$$

24.



$\angle BAM = \alpha$

$DN = r$

$MN = x$

$AN = AM - NM$ **1**

$= h - x$

In ΔADN, $\tan\alpha = \dfrac{r}{h-x}$

Volume of cylinder

$V = \pi r^2 x = \pi(h-x)^2 \tan^2\alpha . x$

$= \pi \tan^2\alpha(x^3 - 2hx^2 + h^2x]$ **1**

$$\frac{dV}{dx} = \pi\tan^2\alpha(3x^2 - 4hx + h^2)$$

$$\frac{d^2V}{dx^2} = \pi \tan^2 \alpha(6x - 4h) \quad 1$$

$$\frac{dV}{dx} = 0$$

$$\Rightarrow \quad 3x^2 - 4hx + h^2 = 0$$

$$(x - h)(3x - h) = 0 \quad \mathbf{1}$$

$$x = h,$$

$$x = \frac{h}{3}$$

$$x = h, \text{ not possible}$$

$$\therefore \quad x = \frac{h}{3}$$

when $\quad x = \frac{h}{3}$ **1**

$$\frac{d^2V}{dx^2} = \pi \tan^2 \alpha(2h - 4h)$$

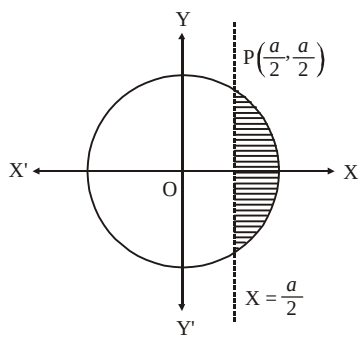
$$= -2\pi h \tan^2 \alpha < 0$$

$\therefore$ V is max, when $x = \frac{h}{3}$ **1**

25. The equations of the given curves are

$$x^2 + y^2 = a^2$$

$$x = \frac{a}{\sqrt{2}}$$



Clearly (1) represents a circle and (2) is the equation of a straight line parallel to y-axis at a distance $\frac{a}{\sqrt{2}}$ units to the right of y-axis. **1**

Solving (1) and (2), we get

$$\frac{a^2}{2} + y^2 = a^2 \Rightarrow y^2 = \frac{a^2}{2}$$

$$y = \frac{a}{\sqrt{2}} \quad \mathbf{1}$$

$\therefore$ P$\left(\frac{a}{\sqrt{2}}, \frac{a}{\sqrt{2}}\right)$ is the point of intersection of curves (1) and (2) in the first quadant.

The smaller region bounded by these two curves is the shaded portion shown in the figure. **1**

$\therefore$ Required area = shaded region

$$= 2\int_{a/\sqrt{2}}^{a} \sqrt{a^2 - x^2}\, dx$$

$$= 2\left[\frac{1}{2}x\sqrt{a^2 - x^2} + \frac{1}{2}a^2 \sin^{-1}\frac{x}{a}\right]_{a/\sqrt{2}}^{a} \quad \mathbf{1}$$

$$= \left[0 - \frac{a}{\sqrt{2}}\sqrt{a^2 - \frac{a^2}{2}} + a^2\left(\sin^{-1} 1 - \sin^{-1}\frac{1}{\sqrt{2}}\right)\right]$$

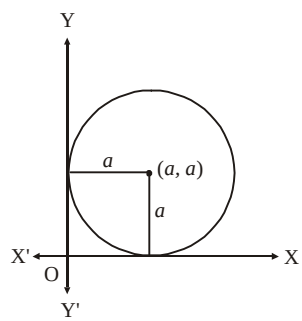
$$= \left[\frac{-a^2}{2} + a^2 \times \frac{\pi}{2} - a^2 \times \frac{\pi}{4}\right]$$

$$= \frac{\pi a^2}{4} - \frac{a^2}{2} = \frac{a^2}{2}\left(\frac{\pi}{2} - 1\right) \text{ sq. units.} \quad \mathbf{1}$$

26. Family of circles which touch the coordinate axes in first quadrant is

$$(x - a)^2 + (y - a)^2 = a^2 \quad \text{...(1)}$$

where a is the radius of the circle.



Differentiating w.r.t. x

$$2(x - a) + 2(y - a)\frac{dy}{dx} = 0$$

Putting $\quad \frac{dy}{dx} = p$ **1**

$$\therefore \quad (x - a) + (y - a)p = 0$$

$$\text{or} \quad x - a + py - pa = 0$$

$$x + py - a(1 + p) = 0$$

$$\therefore \quad a = \frac{x + py}{1 + p} \quad \mathbf{1}$$

Putting value of a in (1)

$$\left(x - \frac{x + py}{1 + p}\right)^2 + \left(y - \frac{x + py}{1 + p}\right)^2 = \left(\frac{x + py}{1 + p}\right)^2$$

$$\left(\frac{x(1 + p) - x - py}{1 + p}\right)^2 + \left(\frac{y(1 + p) - x - py}{1 + p}\right)^2$$

$$= \left(\frac{x+py}{1+p}\right)^2 \quad \textbf{1}$$

or $$\left(\frac{(x-y)p}{1+p}\right)^2 + \left(\frac{(y-x)}{1+p}\right)^2 = \left(\frac{x+py}{1+p}\right)^2$$

Multiplying both sides by $(1+p)^2$

$\therefore\ (x-y)^2 p^2 + (x-y)^2 = (x+py)^2$

or $(x-y)^2 (1+p^2) = (x+py)^2$

$$\Rightarrow (x-y)^2\left(1+\left(\frac{dy}{dx}\right)^2\right) = \left(x+y\frac{dy}{dx}\right)^2$$

This is the required diff. equation. **1**

27. Equation of any plane passing through $(-1, -1, 2)$

$a(x+1) + b(y+1) + c(z-2) = 0$ ...(1)

If plane is $\perp$ to each one of the plane

$2x + 3y - 3z - 2 = 0$

and, $5x - 4y + z - 6 = 0$

$2a + 3b - 3c = 0$ ...(2)

$5a - 4b + c = 0$ ...(3)

On solving (2) and (3), we get

$a = -9k,$

$b = -17k,$

$c = -23k$ **1**

$\therefore\ 9(x+1) + 17(y+1) + 23(z-2) = 0$

$9x + 17y + 23z + 9 + 17 - 46 = 0$

$9x + 17y + 23z - 20 = 0$ **1**

The equation on plane passing through $(-1, -1, 2)$ is

$a(x+1) + b(y+1) + c(z-2) = 0$, where a, b, c are $d.r^n.s$ of normal to the plane

(i) is $\perp$ to $2x + 3y - 3z = 2$ and $5x - 4y + z = 6$

$\therefore\ 2a + 3b - 3c = 0$ **1**

and $5a - 4b + c = 0$

$$\therefore \frac{a}{-9} = \frac{b}{-17} = \frac{c}{-23}$$

The $d.r^n.s$ of normal to the plane are 9, 17, 23

$\Rightarrow$ Equation of plane is **1**

$9(x+1) + 17(y+1) + 23(z-2) = 0$

$\Rightarrow 9x + 17y + 23z = 0$ **1**

OR

Let $P(t^2, 2t)$ be the point on $y^2 = 4x$, nearst to the point A $(2, -8)$, then

$AP^2 = (t^2-2)^2 + (2t+8)^2$ **1**

Taking $y = AP^2$, we have

$y = (t^2-2)^2 + (2t+8)^2$

$= (t^2-2)^2 + (2t+8)^2$

$= (t^4 - 4t^2 + 4) + (4t^2 + 32t + 64)$ **1**

$= t^4 + 32t + + 68$

$$\Rightarrow \frac{dy}{dt} = 4t^3 + 32$$

and $$\frac{d^2y}{dt^2} = 12t^2 \quad \textbf{1}$$

For maximum or minimum

$$\frac{dy}{dx} = 0$$

$\Rightarrow 4(t^3 + 8) = 0$

$\Rightarrow (t+2)(t^2 - 2t + 4) = 0$

$\Rightarrow t = -2$

$$\left.\frac{d^2y}{dt^2}\right|_{t=-2} = +\text{ve} \quad \textbf{1}$$

$\Rightarrow y$ is minimum

Hence AP is minimum. **1**

$\Rightarrow P(t^2, 2t) = (4, -4)$

is the point on $y^2 = 4x$,

nearest to the given point A $(2, -8)$ **1**

28. Let x kg and y kg of fertilizers of type I and II meet the requirement. According to hypothesis, the L.P.P. is

Minimize $Z = 2x + 3y$

Subject to the constraints,

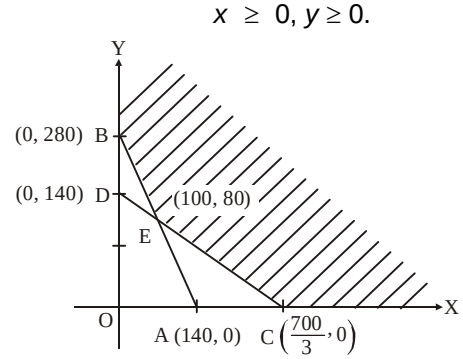
$$x \times \frac{10}{100} + y \times \frac{5}{100} \geq 14 \quad \textbf{1}$$

$\Rightarrow 10x + 5y \geq 1400$

$$x \times \frac{10}{100} + y \times \frac{10}{100} \geq 14 \quad \textbf{1}$$

$\Rightarrow 6x + 10y \geq 1400$

$x \geq 0, y \geq 0.$ **1**



Now, draw the lines

AB : $10x + 5y = 1400$

CD : $6x + 10y = 1400$ **1**

Lines AB and CD meet at E(100, 80).

The feasible region has been shaded. This is an unbounded region vertices C, E and B.

At C $\left(\frac{700}{3}, 0\right)$, the cost is

$$Z = 2 \times \frac{700}{3} + 3 \times 0 = \frac{1400}{3}$$ **1**

At E (100, 80), the cost is

$$Z = 2 \times 100 + 3 \times 80 = 440$$

At B(0, 280), the cost is

$$Z = 2 \times 0 + 3 \times 280 = 840$$

Clearly, 440 is the minimum of the three.

Thus, 100 kg of type I and 80 kg of type II fertilisers should be used to meet the desired requirement and minimum cost. **1**

29. Consider the following events

E = A hits the target.

F = B hits the target

and G = C hits the target

We have, $P(E) = \frac{4}{5}$, $P(F) = \frac{3}{4}$ and $P(G) = \frac{2}{3}$

(i) We have,

Required probability = $P(E \cap F \cap G)$

Required probability = $P(E)\,P(F)\,P(G)$

{$\because$ A,B, C are independent events}

$\Rightarrow$ Required probability = $\frac{4}{5} \times \frac{3}{4} \times \frac{2}{3} = \frac{2}{5}$ **1**

(ii) We have,

Required probability = $P(\overline{E} \cap F \cap G)$

$\Rightarrow$ Required probability = $P(\overline{E})\,P(F)\,P(G)$

[$\because$ A, B, C are independent events]

Required probability = $\left(1 - \frac{4}{5}\right) \times \frac{3}{4} \times \frac{2}{3} = \frac{1}{10}$ **1**

(iii) We have,

Required probability = $P(E \cap F \cap \overline{G})$

$\cup\,(\overline{E} \cap F \cap G) \cup (E \cap \overline{F} \cap G)$ **1**

$\Rightarrow$ Required probability = $P(E)\,P(F)\,P(\overline{G})$

$+ P(\overline{E})\,P(F)\,P(G) + P(E)\,P(\overline{F})\,P(G)$ **1**

$\Rightarrow$ Required probability

$$= \frac{4}{5} \times \frac{3}{4} \times \frac{1}{3} + \frac{1}{5} \times \frac{3}{4} \times \frac{2}{5} \times \frac{4}{5} + \frac{1}{4} \times \frac{2}{3} = \frac{13}{30}$$ **1**

(iv) Required probability = $P(\overline{E} \cap \overline{F} \cap \overline{G})$

$= P(\overline{E})\,P(\overline{F})\,P(\overline{G})$

$= \frac{1}{5} \times \frac{1}{4} \times \frac{1}{3} = \frac{1}{60}$ **1**

●●

Sample Question Paper—10

Section–A

1. For skew symmetric matrix

$$A' = -A$$

$$A = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix},$$

$$A' = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

$$= -A$$

$\because$ $A' = -A$

Hence A is a skew symmetric matrix. **1**

2. A and B are symmetric matrices.

$\Rightarrow$ $A' = A'B' = B$

Now $(AB - BA)' = (AB)' - (BA)'$

$= B'A' - A'B'$

$= BA - AB$

$\because$ $B' = B,\ A' = A$

$= -(AB - BA)$

$\Rightarrow$ AB – BA is a skew symmetric matrix. **1**

Part (A) is the correct answer.

3. Let $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$

$\therefore$ $kA = k\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$

$= \begin{bmatrix} ka_1 & kb_1 & kc_1 \\ ka_2 & kb_2 & kc_2 \\ ka_3 & kb_3 & kc_3 \end{bmatrix}$ **½**

$|kA| = \begin{bmatrix} ka_1 & kb_1 & kc_1 \\ ka_2 & kb_2 & kc_2 \\ ka_3 & kb_3 & kc_3 \end{bmatrix}$

Taking out k common from each row

$= k^3 \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} = k^3\,|A|$ **½**

4. $y = 3e^{2x} + 2e^{3x}$

Diff. w.r.t. x

$$\frac{dy}{dx} = 6e^{2x} + 6e^{3x}$$

Again diff. w.r.t. x

$$\frac{d^2y}{dx^2} = 12e^{2x} + 18e^{3x}$$ **½**

We have $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y$

$= 12e^{2x} + 18e^{3x} - 5(6e^{2x} + 6e^{3x}) + 6(3e^{2x} + 2e^{3x})$

$= 12e^{2x} + 18e^{3x} - 30e^{x} - 30e^{3x} + 18e^{2x} + 12e^{3x}$

$= 0.$ **Proved.** **½**

5. Ellipse is

$$16x^2 + 9y^2 = 400 \quad ...(1)$$

Differentiating (1) w.r.t. x, we get

$$32x + 18y\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{16}{9}\cdot\frac{x}{y} \quad ...(2)$$

By hypothesis,

$$\frac{dy}{dt} = -\frac{dx}{dt}$$

$$\Rightarrow \frac{dy}{dx} = -1 \quad ...(3)$$

$$\therefore \frac{16}{9}\frac{x}{y} = 1, \quad \text{[By (2) and (3)]}$$

$$\Rightarrow x = \frac{9}{16}y \quad ...(4)$$ **½**

$\therefore$ From (1), we get

$$16\left(\frac{9}{16}y\right)^2 + 9y^2 = 400$$

$$\Rightarrow \left(\frac{81}{16} + 9\right)y^2 = 400$$

$$\Rightarrow \quad y^2 = \frac{400 \times 16}{225} = \frac{16 \times 16}{9}$$

$$= \left(\frac{16}{3}\right)^2$$

$$\Rightarrow \quad y = \pm \frac{16}{3}$$

From (4), we get

$$x = \pm 3$$

$\therefore$ At the points $\left(3, \frac{16}{3}\right)$ and $\left(-3, -\frac{16}{3}\right)$ ellipse behaves as desired. **½**

6. $$\int x^2\left(1 - \frac{1}{x^2}\right) dx = \int (x^2 - 1)$$

$$= \int x^2 dx - \int dx$$

$$= \frac{x^3}{3} - x + c \qquad \textbf{1}$$

7. $$y = e^x + 1$$

Differentiating w.r.t. x,

$$y' = e^x + 0 \qquad ...(1)$$

Again differentiating

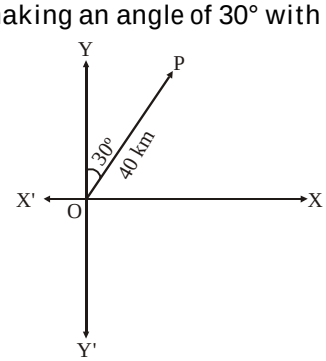
$$y'' = e^x \qquad ...(2)$$

Subtracting (1) and (2),

$$y'' - y' = e^x - e^x = 0$$

$\therefore$ Differential equation is $y'' - y' = 0$. **1**

8. A line segment of 2 cm is drawn on hight of OY making an angle of 30° with it.



OP = 40 km, scale

I cm = 20 km

Vector $\overrightarrow{OP}$ represents displacement of 40 km, 30° east of length. **½**

9. $$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{0.32}{0.5}$$

$$= \frac{32}{50} = \frac{16}{25} \qquad \textbf{1}$$

10. We have,

$$2P(A) = P(B) = \frac{5}{13}$$

$$P(A) = \frac{5}{26}$$

$$P(B) = \frac{5}{13}$$

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

$$\frac{2}{5} = \frac{P(A \cap B)}{\frac{5}{13}}$$

$$P(A \cap B) = \frac{2}{5} \times \frac{5}{13} = \frac{2}{13}$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \qquad \textbf{½}$$

$$= \frac{5}{26} + \frac{5}{13} - \frac{2}{13}$$

$$= \frac{5 + 10 - 4}{26} = \frac{11}{26}$$

$$\Rightarrow \quad P(A \cap B) = \frac{2}{5} \times \frac{5}{13} = \frac{2}{13}$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{5}{26} + \frac{5}{13} - \frac{2}{13}$$

$$= \frac{5 + 10 - 4}{26} = \frac{11}{26} \qquad \textbf{½}$$

Section–B

11. L = Set of all the lines in XY. plane

$R = \{(L_1, L_2) : L_1$ is parallel to $L_2\}$

(i) (a) L_1 is parallel to itself $\Rightarrow$ R is reflexive. **1**

(b) L_1 is parallel to $L_2 \Rightarrow L_2$ is parallel to L_1 $\Rightarrow$ R is symmetric. **1**

(c) Let L_1 is parallel to L_2 and L_2 is parallel to L_3.

Obviously L_1 is parallel to L_3.

$\Rightarrow$ R is transitive. **1**

Hence R is an equivalence relation.

(ii) Set of parallel lines related by $y = 2x + 4$ is $y = 2x + c$, where c is an arbitrary constant. **1**

12. Consider $\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$

$$= \begin{vmatrix} 2(b+c) & 2(c+a) & 2(a+b) \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$$ **1**

[By performing $R_1 \to R_1 + (R_2 + R_3)$]

$$= 2\begin{vmatrix} b+c & c+a & a+b \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$$

[By taking 2 common from R_1]

$$= 2\begin{vmatrix} b+c & c+a & a+b \\ -c & 0 & -a \\ -b & -a & 0 \end{vmatrix}$$

[By performing $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$]

$$= 2\begin{vmatrix} 0 & c & b \\ -c & 0 & -a \\ -b & -a & 0 \end{vmatrix}$$

[By performing $R_1 \to R_1 + (R_2 + R_3)$]

Performing $C_2 \to bC_2$ and $C_3 \to cC_3$ and dividing the determinant by bc, we get

$$= \frac{2}{bc}\begin{vmatrix} 0 & bc & bc \\ -c & 0 & -ac \\ -b & -ab & 0 \end{vmatrix}$$

$$= \frac{2}{bc}\begin{vmatrix} 0 & 0 & bc \\ -c & ab & -ac \\ -b & -ab & 0 \end{vmatrix}$$

[By performing $C_2 \to C_2 - C_3$]

$$= \frac{2}{bc}[bc\,(abc + abc)]$$

$= 4abc$ = R.H.S. **1**

OR

Let $$\Delta = \begin{vmatrix} b+c & c+a & a+b \\ q+r & r+p & p+q \\ y+z & z+x & x+y \end{vmatrix}$$

Operating $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = \begin{vmatrix} 2(a+b+c) & c+a & a+b \\ 2(p+q+r) & r+p & p+q \\ 2(x+y+z) & z+x & x+y \end{vmatrix}$$

$$= 2\begin{vmatrix} a+b+c & c+a & a+b \\ p+q+r & r+p & p+q \\ x+y+z & z+x & x+y \end{vmatrix}$$

Operating $C_2 \to C_2 - C_1$; $C_3 \to C_3 - C_1$, we get

$$= 2\begin{vmatrix} a+b+c & -b & -c \\ p+q+r & -q & -r \\ x+y+z & -y & -z \end{vmatrix}$$

$$= 2(-1)(-1)\begin{vmatrix} a+b+c & b & c \\ p+q+r & q & r \\ x+y+z & y & z \end{vmatrix}$$ **1**

Operating $C_1 \to C_1 - (C_2 + C_3)$, we get

$$\Delta = 2\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$$ **1**

13. We have $y = e^{a\cos^{-1}x}$

$\therefore$ $$y = e^{a\cos^{-1}x}\,\frac{-a}{\sqrt{1-x^2}}$$ **1**

$\Rightarrow$ $$\sqrt{1-x^2}\,y_1 = ae^{a\cos^{-1}x}$$

$\Rightarrow$ $$\sqrt{1-x^2}\,y_2 + y_1 \cdot \frac{1}{2}\cdot\frac{1}{\sqrt{1-x^2}}\,(-2x)$$ **1**

$$= \frac{a^2 e^{a\cos^{-1}x}}{\sqrt{1-x^2}}$$

$\Rightarrow$ $(1 - x^2)y_2 - xy_1 = a^2y$

$\Rightarrow$ $(1 - x^2)y_2 - xy_1 - a^2y = 0.$ **1**

$\Rightarrow$ $$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0.$$ **Proved**

14. The given equation may be written as

$$x\sqrt{1+y} = -y\sqrt{1+x}$$

Squaring both sides, we get

$x^2(1 + y) = y^2(1 + x)$ **1**

$\Rightarrow$ $x^2 - y^2 = y^2x - x^2y$

$\Rightarrow$ $(x + y)(x - y) = -xy(x - y)$

$\Rightarrow$ $x + y = -xy$, $[\because x - y \neq 0]$

$\Rightarrow$ $x = -y - xy$ **1**

$\Rightarrow$ $y(1 + x) = -x$

$\Rightarrow$ $$y = -\frac{x}{1+x}$$

$\therefore$ $$\frac{dy}{dx} = -\left\{\frac{(1+x).1 - x(0+1)}{(1+x)^2}\right\}$$ **1**

$\Rightarrow$ $$\frac{dy}{dx} = -\frac{1}{(1+x)^2}$$ **1**

15. $x = \sin 3t,\ y = \cos 2t$

$$\frac{dx}{dt} = 3\cos 3t$$

$$\frac{dy}{dt} = -2\sin 2t$$

$$\frac{dy}{dt} = \frac{dy}{dt}\cdot\frac{dt}{dx} \quad \textbf{1}$$

$$= \frac{-2\sin 2t}{3\cos 3t}$$

$$\left(\frac{dy}{dx}\right)_{t=\frac{\pi}{4}} = \frac{-2\sin 2.\frac{\pi}{4}}{3\cos 3.\frac{\pi}{4}} \quad \textbf{1}$$

$$= \frac{-2\sin\frac{\pi}{4}}{3\cos 3\frac{\pi}{4}}$$

$$= \frac{-2\times 1}{-3\times\frac{1}{\sqrt{2}}}$$

$$= \frac{2\sqrt{2}}{3} \quad \textbf{1}$$

$$(y - \cos 2t) = \frac{2\sqrt{2}}{3}(x - \sin 3t)$$

$$y - \cos 2.\frac{\pi}{4} = \frac{2\sqrt{2}}{3}\left(x - \sin\frac{3\pi}{4}\right)$$

$$y = \frac{2\sqrt{2}}{3}\left(x - \frac{1}{\sqrt{2}}\right)$$

$$= \frac{2\sqrt{2}}{3}\left(\frac{\sqrt{2}x - 1}{\sqrt{2}}\right)$$

$$3\sqrt{2}y = 4x - 2\sqrt{2} \quad \textbf{1}$$

$$4x - 3\sqrt{2}y - 2\sqrt{2} = 0.$$

OR

Let at point (x, y) tangent is parallel to x-axis.

$\because\ y = 2x^3 - 15x^2 + 36x - 21$ ...(i)

$\because$ Slope of tangent

$$= \left(\frac{dy}{dx}\right)_{(x,y)} \quad \textbf{1}$$

$$= 6x^2 - 30x + 36$$

$\because$ tangent is parallel to x-axis

$\therefore\ m_{\text{tangent}} = m_{x\text{-axis}}$

$\Rightarrow \left(\frac{dy}{dx}\right)_{(x,y)} = 0$

$\Rightarrow\ 6x^2 - 30x + 36 = 0$

$\Rightarrow\ 6(x^2 - 5x + 6) = 0$

$\Rightarrow\ (x - 2)(x - 3) = 0$

$\Rightarrow\ x = 2,\ x = 3$ **1**

Putting these values of x in equation (i),

Putting $x = 2$,

$y = 2(2)^3 - 15(2)^2 + 36(2) - 21$

$= 16 - 60 + 72 - 21$

$= 7$

Putting $x = 3$,

$y = 2(3)^3 - 15(3)^2 + 36(3) - 21$

$= 54 - 135 + 108 - 21$ **1**

$= 16$

$\therefore$ At points (2, 7) and (3, 16) the tangents are parallel to x-axis.

Equation of tangent at (2, 7) is

$y - y_1 = m(x - x_1)$

$\Rightarrow\ (y - 7) = 0\,(x - 2)$

$\Rightarrow\ y = 7$

Equation of tangent at (3, 16) is

$y - y_1 = m(x - x_1)$

$y - 16 = 0\,(x - 3)$ **1**

$\Rightarrow\ y = 16$

16. $$I = \int \sin x\sqrt{1 + (2\cos^2 x - 1)}\,dx$$

$$= \int \sin x.\sqrt{2}\cos x\,dx \quad \textbf{1}$$

$$= \sqrt{2}\int \sin x\cos x\,dx \quad \textbf{1}$$

Let $\sin x = t \Rightarrow \cos x\,dx = dt$

$$I = \sqrt{2}\int t\,dt + c = \sqrt{2}.\frac{t^2}{2} + c \quad \textbf{1}$$

$$= \frac{t^2}{\sqrt{2}} + c = \frac{\sin^2 x}{\sqrt{2}} + c \quad \textbf{1}$$

OR

$$I = \int \frac{x^2 + x + 1}{(x+2)(x+1)^2}\,dx$$

Let $\int \frac{x^2 + x + 1}{(x+2)(x+1)^2}\,dx =$

$$\frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x+2} \quad \textbf{1}$$

Comparing the numerators both sides, we have

$(x^2 + x + 1) = A\,(x + 1)\,(x + 2)$

$+ B(x + 2) + C\,(x + 1)^2$ ...(1)

Putting $x = -1$ in (1), we get

$1 - 1 + 1 = A(0)\,(1) + B(1) + C(0)^2$

$B = 1$ ...(2) **1**

Putting $x = -2$ in (1), we get

$$4 - 2 + 1 = A(-1)(0) + B(0) + C(-1)^2$$

$$3 = C$$

Comparing the coefficients of x^2 both sides of (1)

$$1 = A + C \Rightarrow A = 1 - C = 1 - 3 \quad \textbf{1}$$

$$A = -2$$

Thus, $I = \int \frac{-2}{x+1}dx + \int \frac{dx}{(x+1)^2} + \int \frac{-3}{(x+2)}dx$

$$= -2\log|x+1| - \frac{1}{x+1} + 3\log|x+2| + c. \quad \textbf{1}$$

17. $\int \frac{dx}{x[6(\log x)^2 + 7(\log x) + 2]} = \int \frac{dt}{6t^2 + 7t + 2}$

[Let $\log x = t \Rightarrow \frac{1}{x}dx = dt$]

$$= \frac{1}{6}\int \frac{dt}{t^2 + \frac{7}{6} + \frac{1}{3}} \quad \textbf{1}$$

$$= \frac{1}{6}\int \frac{dt}{\left(t + \frac{7}{12}\right)^2 - \frac{49}{144} + \frac{1}{3}}$$

$$= \frac{1}{6}\int \frac{dt}{\left(t + \frac{7}{12}\right)^2 - \frac{1}{144}} \quad \textbf{1}$$

$$= \frac{1}{6}\int \frac{dt}{\left(t + \frac{7}{12}\right)^2 - \left(\frac{1}{12}\right)^2}$$

$$= \frac{1}{6}\left[\frac{1}{2 \times \frac{1}{12}}\log\left|\frac{t + \frac{7}{12} - \frac{1}{12}}{t + \frac{7}{12} + \frac{1}{12}}\right|\right] + c \quad \textbf{1}$$

$$= \log\left|\frac{t + \frac{1}{2}}{t + \frac{2}{3}}\right| + c$$

$$= \log\left|\frac{\log x + \frac{1}{2}}{\log x + \frac{2}{3}}\right| + c \quad \textbf{1}$$

18. $I = \int \tan^{-1}\sqrt{\frac{1 - \sin x}{1 + \sin x}}dx$

Now, $\sqrt{\frac{1 - \sin x}{1 + \sin x}} = \frac{\sqrt{\left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)^2}}{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)^2}$ **1**

$$= \frac{\cos\frac{x}{2} - \sin\frac{x}{2}}{\cos\frac{x}{2} + \sin\frac{x}{2}} \quad \textbf{1}$$

$$= \frac{1 - \tan\frac{x}{2}}{1 + \tan\frac{x}{2}}$$

$$= \tan\left(\frac{\pi}{4} - \frac{x}{2}\right)$$

$$I = \int \tan^{-1}\left(\tan\left(\frac{\pi}{4} - \frac{x}{2}\right)\right)dx \quad \textbf{1}$$

$$= \int \left(\frac{\pi}{4} - \frac{x}{2}\right)dx$$

$$= \frac{\pi}{4}x - \frac{x^2}{4} + c \quad \textbf{1}$$

19. Given, $x(1 + y^2)\,dx - y(1 + x^2)\,dy = 0$

$\Rightarrow \quad x(1 + y^2)\,dx = y(1 + x^2)\,dy$

$$\frac{x}{1 + x^2}dx = \frac{y}{1 + y^2}dy \quad \textbf{1}$$

Integrating both sides,

$$\int \frac{x}{1 + x^2}dx = \int \frac{y}{1 + y^2}dy$$

Let $1 + x^2 = t$ and $1 + y^2 = u$

$dx\,(2x) = dt$, $2y\,dy = du$

$\therefore$ Given equation becomes

$$\frac{1}{2}\int \frac{dt}{t} = \frac{1}{2}\int \frac{du}{u} \quad \textbf{1}$$

$$\log(t) = \log(u) + C$$

$$\log(1 + x^2) = \log(1 + y^2) + C$$

Given $y = 0$ at $x = 1$

$$\log(1 + 1) = \log 1 + C$$

$$\log 2 = C$$

$\therefore$ Required equation becomes

$$\log(1 + x^2) = \log(1 + y^2) + \log 2. \quad \textbf{1}$$

OR

The given differential equation is

$$\frac{dy}{dx} + \sec x.y = \tan x \quad ...(1)$$

This is a linear differential equation.

Here, $P = \sec x$ and $Q = \tan x$

Now, $\text{I.F.} = e^{\int P\,dx} = e^{\int \sec x\,dx}$

$= \log(\sec x + \tan x)$

$= \sec x + \tan x$ **1**

$\therefore$ The solution of (1) is,

$y.(\sec x + \tan x)$

$= \int \tan x(\sec x + \tan x)\,dx + C$ **1**

$= \int (\sec x \tan x + \tan^2 x)\,dx + C$

$= \int \sec x \tan x\,dx + \int \sec^2 x\,dx - \int 1\,dx + C$

$= \sec x + \tan x - x + C.$ **1**

20. We know that

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

Here $\vec{a}\cdot\vec{b} = \frac{1}{2}$, $|\vec{a}| = |\vec{b}|$

and $\theta = 60^\circ$ **1**

$\therefore \quad \frac{1}{2} = |\vec{a}|\,|\vec{b}|\cos 60^\circ$

$\Rightarrow \quad \frac{1}{2} = |\vec{a}|^2\left(\frac{1}{2}\right)$ **1**

$\Rightarrow \quad |\vec{a}|^2 = 1$

$[\because |\vec{a}| = |\vec{b}|$, given$]$

$\Rightarrow \quad |\vec{a}| = 1$

$|\vec{b}| = |\vec{a}| = 1$ **1**

Thus, $|\vec{a}| = 1, |\vec{b}| = 1$ **1**

21. Let E_1, E_2 and A be the events defined as follows :

E_1 = one occurs, E_2 = one does not occur and A = the man reports that it is a one

We have,

$P(E_1) = \frac{1}{6}$, $P(E_2) = 1 - \frac{1}{6} = \frac{5}{6}$ **1**

$P\left(\frac{A}{E_1}\right)$ = Probability that the man speaks truth $= \frac{3}{5}$

and $P\left(\frac{A}{E_2}\right)$ = Probability that the man does not speak truth

$= 1 - \frac{3}{5} = \frac{2}{5}$

We have to find $P\left(\frac{E_1}{A}\right)$ *i.e.*, the probability that there is 1 on the given that the man has reported that there is 1. By Baye's theorem

$$P\left(\frac{E_1}{A}\right) = \frac{P(E_1)\times P\left(\frac{A}{E_1}\right)}{P(E_1)\times P\left(\frac{A}{E_1}\right) + P(E_2)\times P\left(\frac{A}{E_2}\right)}$$ **1**

$$= \frac{\frac{1}{6}\times\frac{3}{5}}{\frac{1}{6}\times\frac{3}{5}+\frac{5}{6}\times\frac{2}{5}}$$

$= \frac{3}{13}$ **1**

22. Let a fair coins tossed n times.

Pobability of getting a head $= \frac{1}{2}$

Probability of getting no head $= \left(\frac{1}{2}\right)^n$ **1**

$\therefore$ Probability of getting at least one head

$= 1 - \left(\frac{1}{2}\right)^n$ ...(1)

We have probability of getting at least one head

$80\% = 0.8$ ...(2)

From (1) and (2) **1**

$1 - \left(\frac{1}{2}\right)^n > 0.8$

or $\left(\frac{1}{2}\right)^n < 1 - 0.8$ **1**

or $\left(\frac{1}{2}\right)^n < 0.2 \Rightarrow n \geq 3.$ **1**

Section–C

23. $f : A \to B'$ where A = R – {3}, B = R – {1}, f is defined by

$$f(x) = \frac{x-2}{x-3}$$

(a) $f(x_1) = \frac{x_1-2}{x_1-3}$, $f(x_2) = \frac{x_2-2}{x_2-3}$ **1**

$f(x_1) = f(x_2)$

$\Rightarrow \quad \frac{x_1-2}{x_1-3} = \frac{x_2-2}{x_2-3}$ **1**

or $(x_1-2)(x_2-3) = (x_2-2)(x_1-3)$

or $x_1x_2 - 3x_1 - 2x_2 + 6 = x_1x_2 - 2x_1 - 3x_2 + 6$

i.e., $-3x_1 - 2x_2 = -2x_1 - 3x_2$

or $-x_1 = -x_2$ **1**

$\Rightarrow \quad x_1 = x_2$ **1**

$\therefore$ f is one-one.

(b) Let $y = \frac{x-2}{x-3}$, $xy - 3y = x - 2$

or $x(y-1) = 3y - 2 \Rightarrow x = \frac{3y-2}{y-1}$ **1**

$\Rightarrow$ For every value of y except $y = 1$, there is a pre-image.

$$x = \frac{3y-2}{y-1} \qquad \textbf{1}$$

$\Rightarrow$ f is onto.

Thus, f is one-one and onto.

24. The given equations are

$$2x - y + 3z = 5 \qquad ...(1)$$
$$3x + 2y - z = 7 \qquad ...(2)$$
$$4x + 5y - 5z = 9 \qquad ...(3)$$

$\Rightarrow$ $AX = B$

where $A = \begin{bmatrix} 2 & -1 & 3 \\ 3 & 2 & -1 \\ 4 & 5 & -5 \end{bmatrix}$; $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

and $B = \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix}$ **1**

Now $|A| = \begin{vmatrix} 2 & -1 & 3 \\ 3 & 2 & -1 \\ 4 & 5 & -5 \end{vmatrix}$

$= 2(-10+5) + 1(-15+4) + 3(15-8)$ **1**

$= -10 - 11 + 21 = 0.$

Now $\text{adj}A = \begin{bmatrix} -5 & 10 & -5 \\ 11 & -22 & 11 \\ 7 & -14 & 7 \end{bmatrix}$

$\therefore$ $(\text{adj } A).B = \begin{bmatrix} -5 & 10 & -5 \\ 11 & -22 & 11 \\ 7 & -14 & 7 \end{bmatrix} \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix}$ **1**

$= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0$ **1**

$\Rightarrow$ The given system of equations is consistent with infinitely many solutions.

Let $z = k$

Equation (1) $\Rightarrow 2x - y = 5 - 3k$

Equation (2) $\Rightarrow 3x + 2y = 7 + k$

Equation (3) $\Rightarrow 4x + 5y = 9 + 5k$

Solving first two equations, we get

$$x = \frac{1}{7}(17 - 5k) \qquad \textbf{½}$$

and $$y = \frac{1}{7}(-1 + 11k)$$

These values of x and y clearly satisfies the third equation.

$$\therefore \quad x = \frac{1}{7}(17 - 5k)$$
$$y = \frac{1}{7}(-1 + 11k)$$
$$z = k, (k \in R) \qquad \textbf{1}$$

is the required solution.

25. Let x cm and y cm be the sides of the right angled triangle, then

$$x^2 + y^2 = 5^2$$

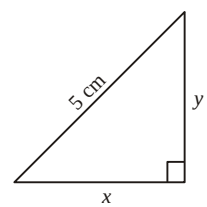
$$\Rightarrow \quad y = \sqrt{25 - x^2}$$

Now, $$A = \frac{1}{2} x \times y$$
$$= \frac{1}{2} x\sqrt{25 - x^2}$$

$$\Rightarrow \frac{dA}{dx} = \frac{1}{2}\left[\sqrt{25 - x^2} - \frac{x^2}{\sqrt{25 - x^2}}\right] \qquad \textbf{1}$$

For maxima or minima,

$$\frac{dA}{dx} = 0 \qquad \textbf{1}$$



$$\Rightarrow \sqrt{25 - x^2} - \frac{x^2}{\sqrt{25 - x^2}} = 0 \qquad \textbf{1}$$

$$\Rightarrow \quad 25 - 2x^2 = 0$$

$$\Rightarrow \quad x^2 = \frac{25}{2}$$

$$\Rightarrow \quad x = \frac{5}{\sqrt{2}} \qquad \textbf{1}$$

Again,

$$\frac{d^2A}{dx^2} = \frac{1}{2}\left[\frac{-x}{\sqrt{25 - x^2}} - \frac{2x}{\sqrt{25 - x^2}} - \frac{x^3}{(25 - x^2)^{3/2}}\right]$$

Clearly, for $x = \frac{5}{\sqrt{2}}$, $\frac{d^2A}{dx^2} < 0$ **1**

$\therefore$ A is maximum and its maximum value is

$$= \frac{1}{2} \cdot \frac{5}{\sqrt{2}} \cdot \sqrt{25 - \frac{25}{2}}$$

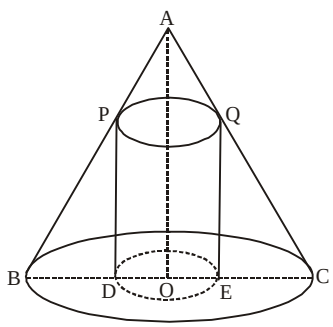
$$= \frac{25}{4} \text{ square units.} \qquad \textbf{1}$$

26. Let r = OC be the radius of the cone and h = OA be its height. Let a cylinder with radius of base circle OE = x be inscribed in the given cone. The height QE of the cylinder is given

$$\frac{QE}{QA} = \frac{EC}{OC} \quad \text{(since } \Delta QEC \sim \Delta AOC)$$

$$\Rightarrow \quad \frac{QE}{h} = \frac{r-x}{r}$$

$$\Rightarrow \quad QE = \frac{h(r-x)}{r} \qquad \mathbf{1}$$



Let S = $S_{(x)}$ be the curved surface of the cylinder, then

$$S_{(x)} = \frac{2\pi h}{r}(rx - x^2)$$

$$\Rightarrow S'_{(x)} = \frac{2\pi h}{r}(r - 2x) \qquad \mathbf{1}$$

$$S''_{(x)} = -\frac{4\pi h}{r}$$

Now $\quad S'_{(x)} = 0$ gives $x = \frac{\pi}{2}$ **1**

Since $S''_{(x)} < 0$ for all x, and in particular at $x = \frac{r}{2}$, **1**

it follows that $x = \frac{r}{2}$ is a point of maxima of S. Hence, the radius of cylinder of greatest curved surface are which can be inscribed in a given cone is half of that of the cone. **1**

27. The given lines can be written as

$$\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 4\hat{k})$$

and $\vec{r} = (2\hat{i} + 4\hat{j} + 5\hat{k}) + (3\hat{i} + 4\hat{j} + 5\hat{k})$

Here, $\vec{a_1} = \hat{i} + 2\hat{j} + 3\hat{k}$ **1**

$\vec{b_1} = 2\hat{i} + 3\hat{j} + 4\hat{k}$

$\vec{a_2} = 2\hat{i} + 4\hat{j} + 5\hat{k}$

$\vec{b_2} = 3\hat{i} + 4\hat{j} + 5\hat{k}$ **1**

Now, $\vec{a_2} - \vec{a_1} = \hat{i} + 2\hat{j} + 2\hat{k}$

and $$\vec{b_1} \times \vec{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix}$$

$$= -\hat{i} + 2\hat{j} - \hat{k}$$

$\therefore \quad |\vec{b_1} \times \vec{b_2}| = \sqrt{1+4+1} = \sqrt{6}$ **1**

and $(\vec{b_1} \times \vec{b_2}).(\vec{a_2} - \vec{a_1})$ **1**

$= (-\hat{i} + 2\hat{j} - \hat{k}).(\hat{i} + 2\hat{j} + 2\hat{k})$

$= -1 + 4 - 2 = 1$

$\therefore$ The S.D. between the given lines

$$= \left| \frac{(\vec{b_1} \times \vec{b_2}).(\vec{a_2} - \vec{a_1})}{|\vec{b_1} \times \vec{b_2}|} \right|$$

$$= \left| \frac{1}{\sqrt{6}} \right| = \frac{1}{\sqrt{6}} \qquad \mathbf{1}$$

28. The two planes are

$4x + 8y + z - 8 = 0$ ...(1)

$y + z - 4 = 0$ ...(2)

The direction ratio of normal of the planes (1) are 4, 8, 1.

The direction ratio of normal of the planes (2) are 0, 1, 1.

These planes are not parallel. **1**

Since, $\quad \frac{4}{0} \neq \frac{8}{1} \neq \frac{1}{1}$

The planes are not perpendicular.

Since $a_1a_2 + b_1b_2 + c_1c_2 = 4.0 + 8.1 + 1.1 \neq 0$ **1**

$\therefore$ Planes are not perpendicular.

The angle θ between them is given by

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}} \qquad \mathbf{1}$$

$$= \frac{4.0 + 8.1 + 1.1}{\sqrt{4^2 + 8^2 + 1^2}\sqrt{0 + 1^2 + 1^2}}$$

$$= \frac{8+1}{\sqrt{16+64+1}\sqrt{1+1}}$$ **1**

$$= \frac{9}{\sqrt{18}\sqrt{2}} = \frac{9}{9\sqrt{2}} = \frac{1}{\sqrt{2}}$$ **1**

$\Rightarrow \theta = \frac{\pi}{4}$

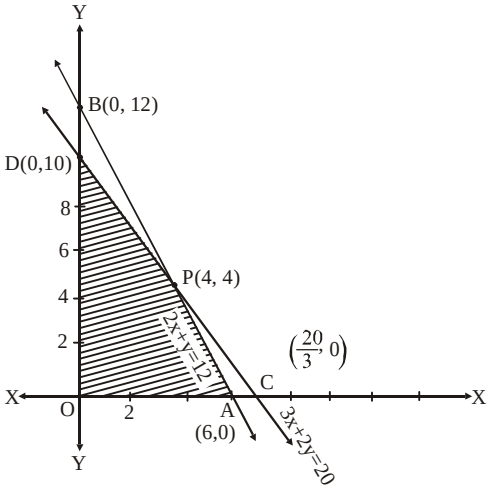
29. Let the manufacturer produces x padestal lamps and y wooden shades; then the time taken by x padestal lamps and y wooden shades on grinding/cutting machines = $(2x + y)$ hours and time taken by x padestal lamps and y shades on the sprayer = $(3x + 2y)$ hours.

Since grinding/cutting machine is available for at the most 12 hours, $2x + y \le 12$ and sprayer is available for at the most 20 hours. We have,

$$3x + 2y \le 20$$

Profit from the sale of x lamps and y shades $Z = 5x + 3y$. **1**

So, our problem is to maximize $Z = 5x + 3y$ subjects to constraints $3x + 2y \le 20$, $2x + y \le 12$, $x, y \ge 0$. Plot the equations $3x + 2y = 20$, $2x + y = 12$ on a graph paper. **1**



(i) The line $2x + y = 12$ passes through A(6, 0) and B(0, 12)

Putting $x = 0$, $y = 0$ in $2x + y \le 12$, we get $0 \le 12$, which is true.

$\Rightarrow$ $2x + y \le 12$ lies on and below AB.

(ii) The line $3x + 2y = 20$ passes through C $\left(\frac{20}{3}, 0\right)$ and D(0, 10). **1**

Putting $x = 0$, $y = 0$ in $3x + 2y \le 20$, we get $0 \le 20$, which is true.

$\Rightarrow$ The region $3x + 2y \le 20$ lies on and below CD.

(iii) $x \ge 0$, is the region on and to the right of y-axis.

(iv) $y \ge 0$. is the region on and above x-axis

The shaded area OAPD is the feasible region where P is intersection of AB and CD.

Now, AB : $2x + y = 12$ ...(1)

CD : $3x + 2y = 20$ ...(2) **1**

Multiplying (1) by 2 and subtract (2) from it

$x = 24 - 20 = 4$

From (1), $y = 12 - 2x = 12 - 8 = 4$

$\therefore$ Point P is (4, 4).

Now, $Z = 5x + 3y$

At A(6, 0) $Z = 30$

At P(4, 4) $Z = 20 + 12 = 32$

At D(0, 10) $Z = 0 + 30$

Thus, the manufacturer should produce 4 lamps at 4 shades to get a maximum profit of Rs. 32. **1**

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